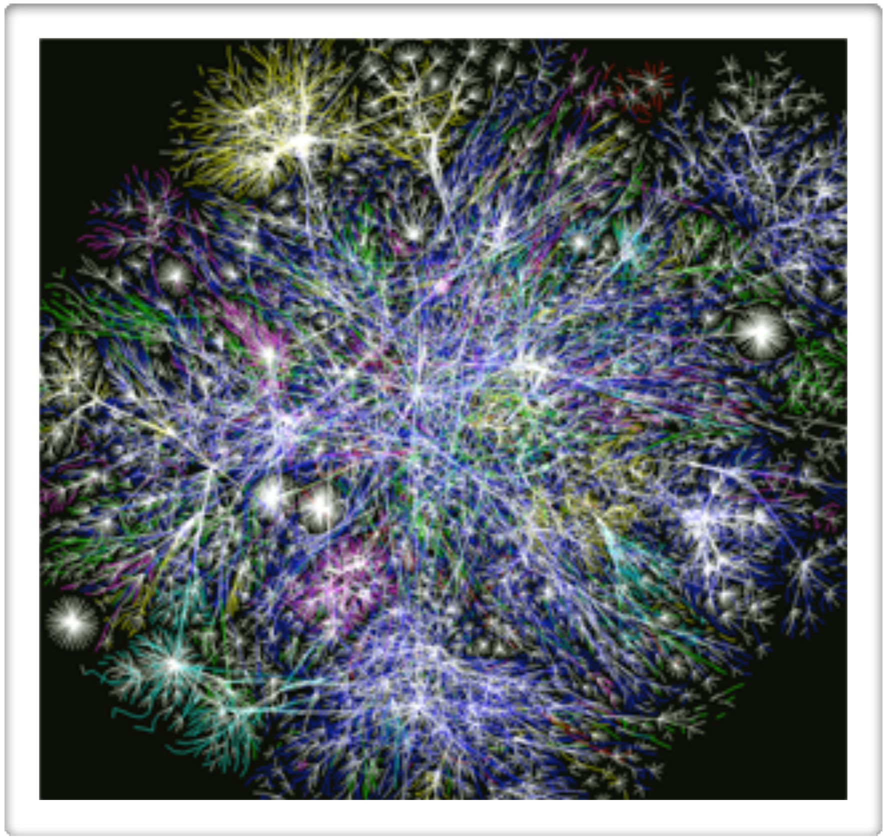
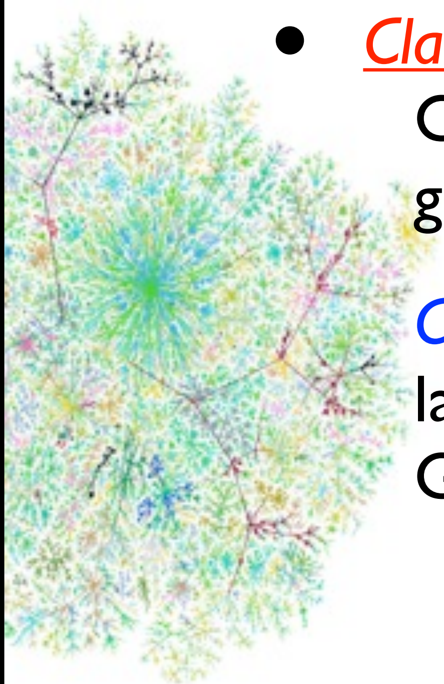
Graph Synopses, Sketches, *and* Streams: A Survey



Sudipto Guha
University of Pennsylvania

Andrew McGregor
University of Massachusetts

Massive Graphs



- *Classic Big Graphs:*
  Call graph (5×10^8 nodes), web graph (5×10^{10} nodes), IP graph (2^{32} nodes), social networks (10^9 nodes), ...

  Challenge: Can't use conventional algorithms on graphs this large. Sometimes can't even store graph in memory! Graphs may be dynamic and/or distributed.

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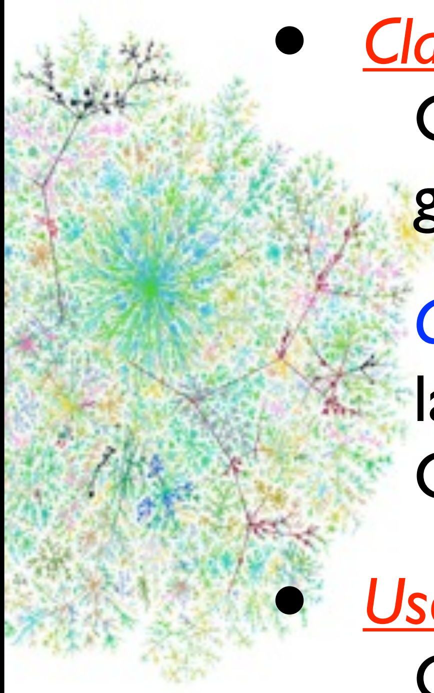

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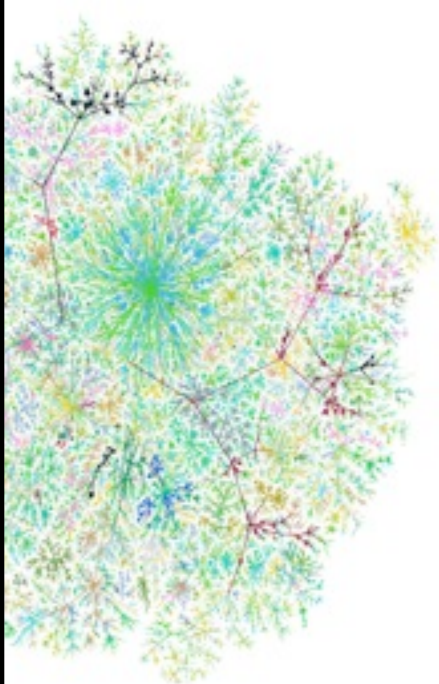
  Challenge: Can't use conventional algorithms on graphs this large. Sometimes can't even store graph in memory! Graphs may be dynamic and/or distributed.

- *Use Abstraction of Structure:*
  Graphs are a natural way to encode structural information where we have data about both basic entities and their relationships. Examples include graphical networks, citation networks, protein interaction and metabolic networks, ...

Focus of Tutorial



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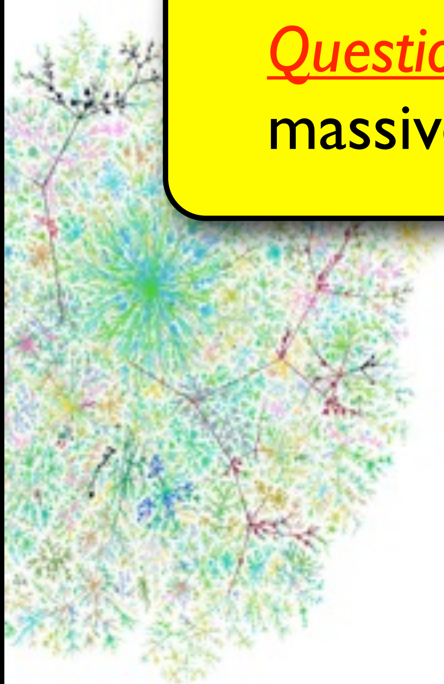

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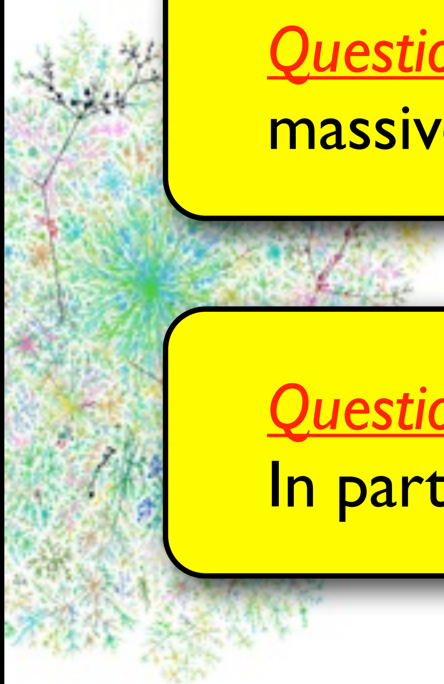

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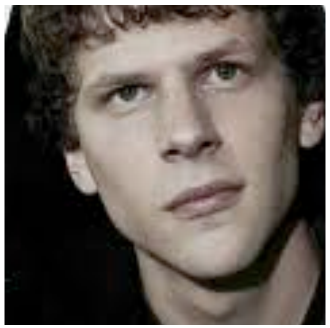
Question 2: How can we construct these synopses efficiently? In particular, what is the input is streaming or distributed?

- Tutorial focuses on the algorithmic and theoretical issues. Consider arbitrary graphs rather than being domain specific.

  This Talk: Definitions & Basic Building Blocks

  Next Talk: Applications & Extensions



[Mark]() and [Erica]() are now friends.

Like · Add Friend

Mark and Erica are no longer friends.

 Like · Add Friend

Eduardo and Mark are now friends.

Like · Add Friend

Tyler and Cameron are friends with Mark.

Like · Add Friend

Sean and Mark are now friends.

Like · Add Friend

Eduardo and Mark are no longer friends.

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Tyler and Cameron are no longer friends with Mark.

Like · Add Friend

Lawyers are now friends with everyone.

Like · Add Friend

Data Streams

- ***Input:*** Observe stream of edges on n nodes added/deleted.

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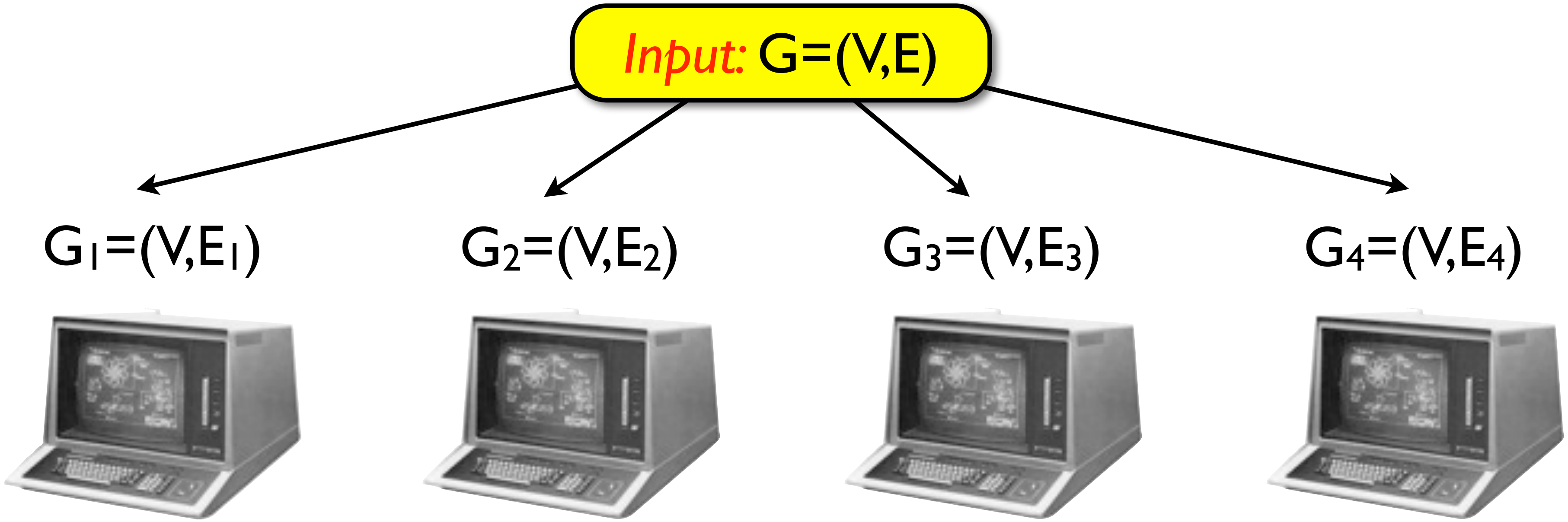
- *Example:* Using $\tilde{O}(n)$ space, maintain connected components.
- *Other Results:* Dense subgraphs, matchings, distances, clustering, partitioning and cuts, diameter, random walks, ...

e.g., [Feigenbaum, Kannan, McGregor, Suri, Zhang 2004, 2005], [McGregor 2005]
[Jowhari, Ghodsi 2005], [Zelke 2008], [Sarma, Gollapudi, Panigrahy 2008, 2009]
[Eggert, Kliemann, Srivastav 2009], [Epstein, Levin, Mestre, Segev 2009]
[Ahn, Guha 2009, 2011], [Kelner, Levine 2011], [Goel, Kapralov, Khanna 2012]

Distributed Processing

Input: G=(V,E)

Distributed Processing



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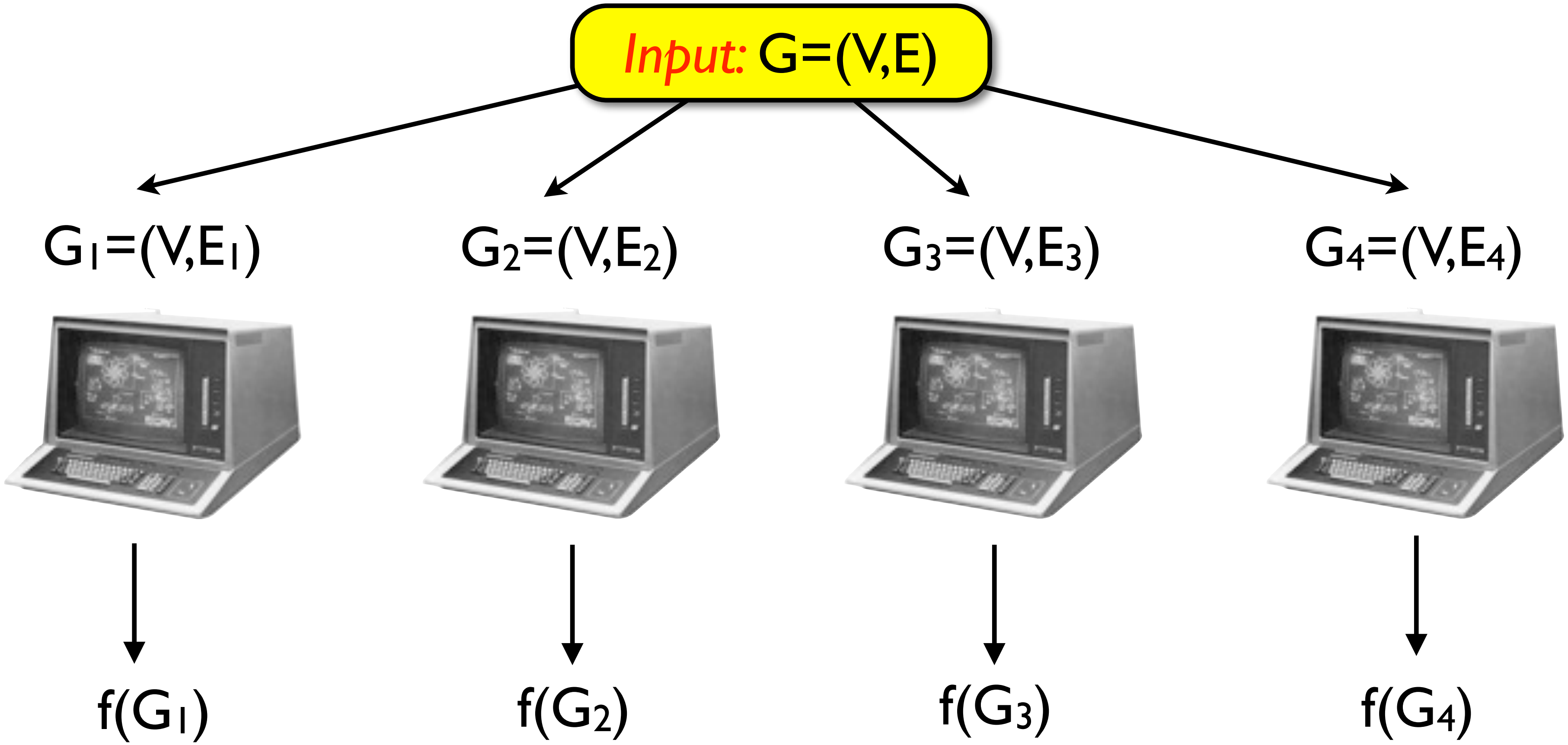

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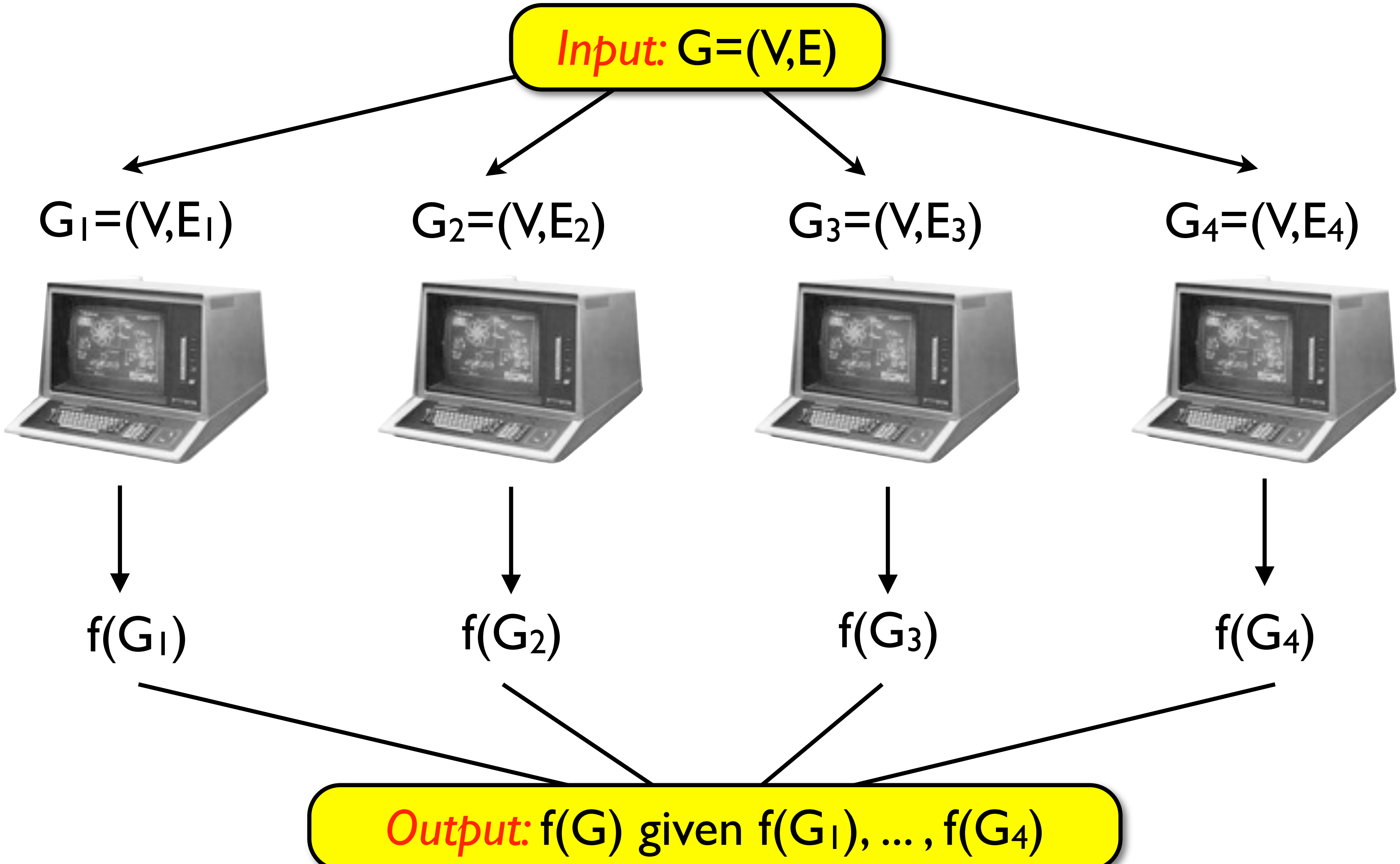
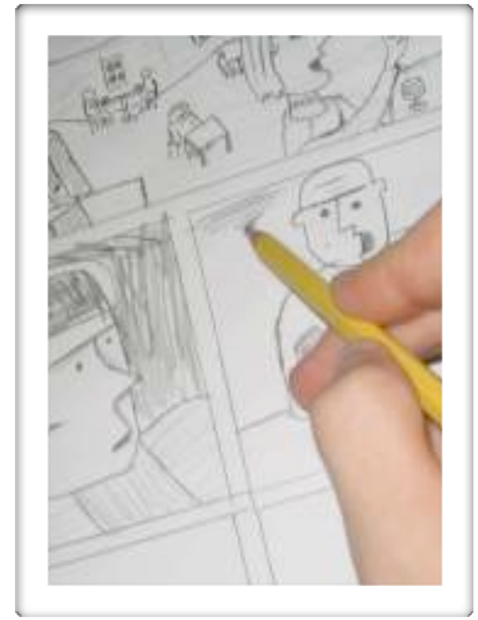


I. **Spanners**

II. **Sparsifiers**



III. **Sketches**

1. **Spanners**

Synopsis for Distance Estimation
"Greedy" Stream Algorithm
Extensions

Spanners & Distances

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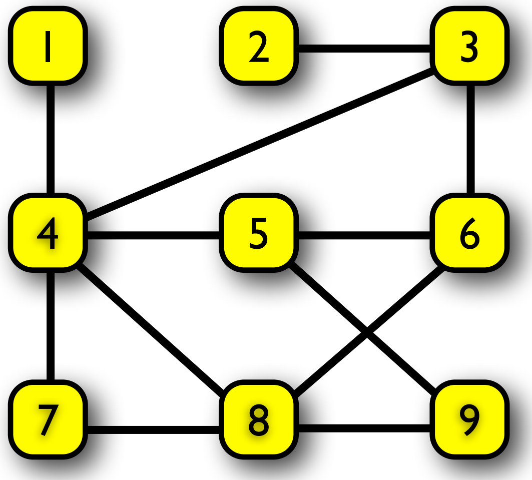
- *Measure:* The distance $d_G(u,v)$ between two nodes u, v is the length of the shortest path between the nodes.



Original Graph G

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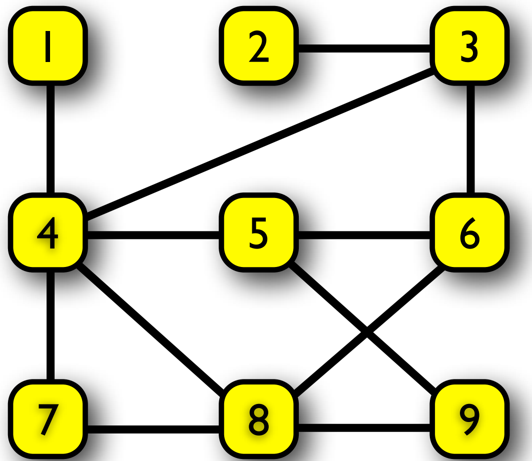

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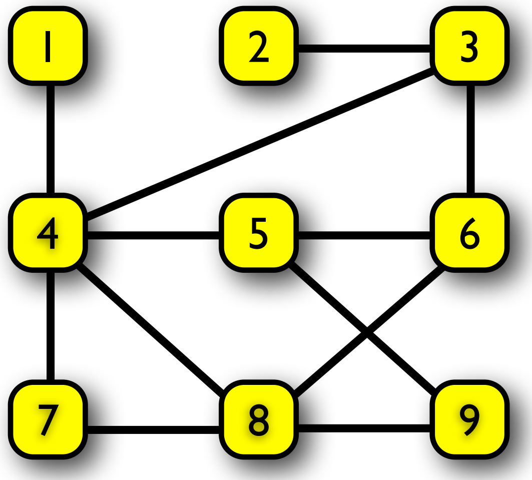

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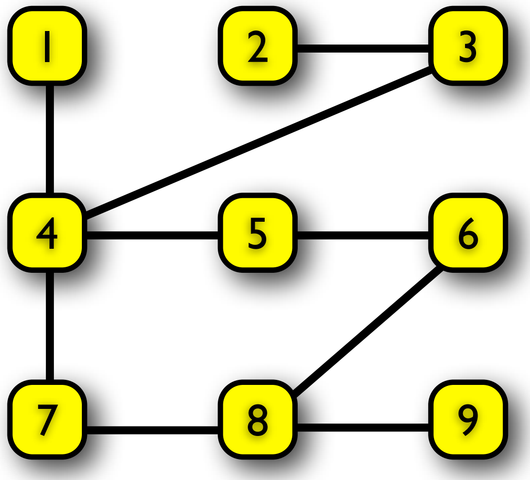

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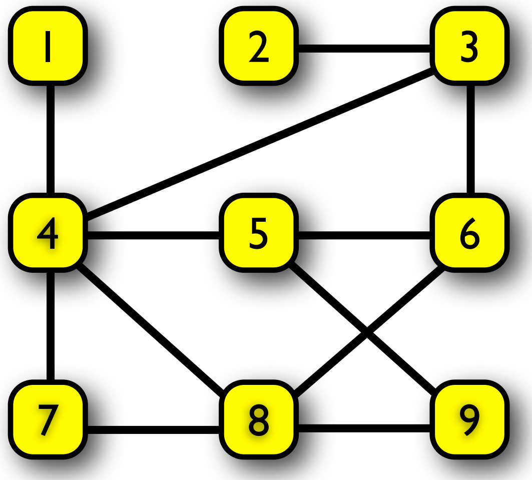

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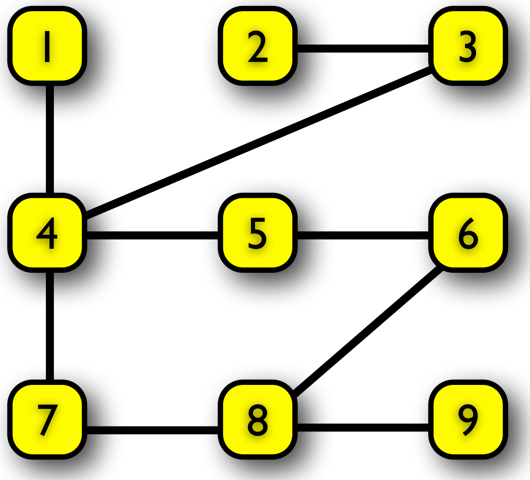

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- *Thm:* Streaming construction using $O(n^{1+2/(k+1)})$ space.

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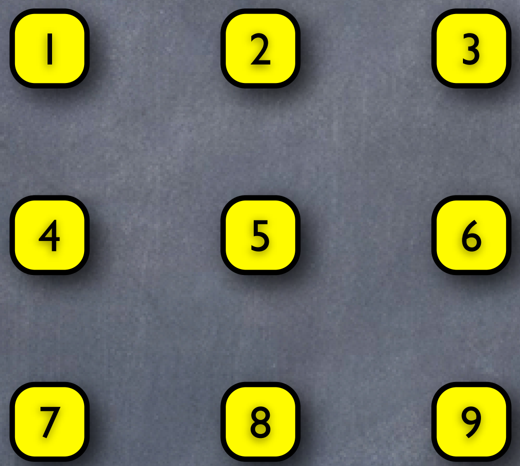

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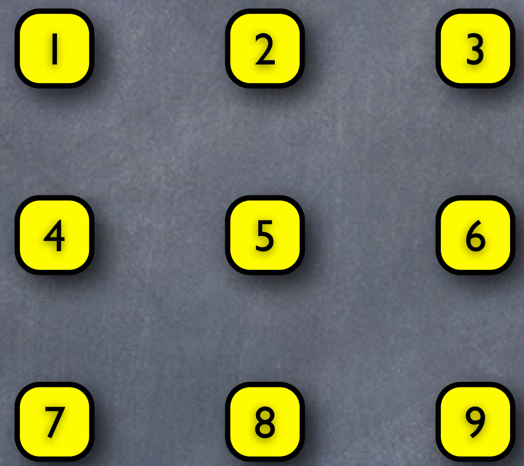

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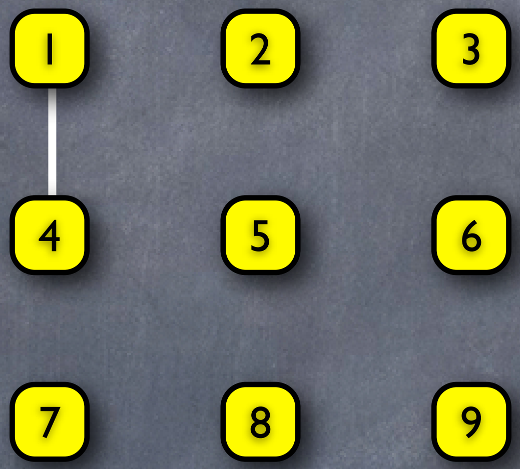

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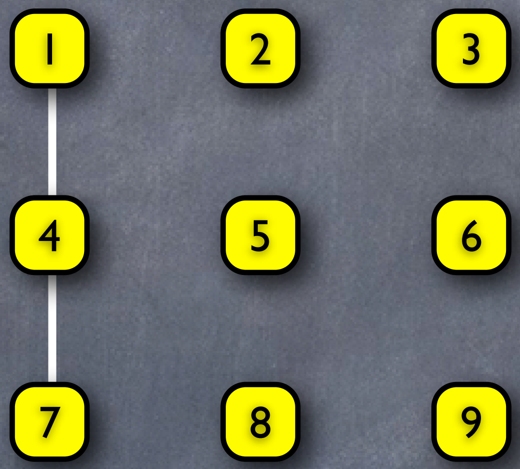

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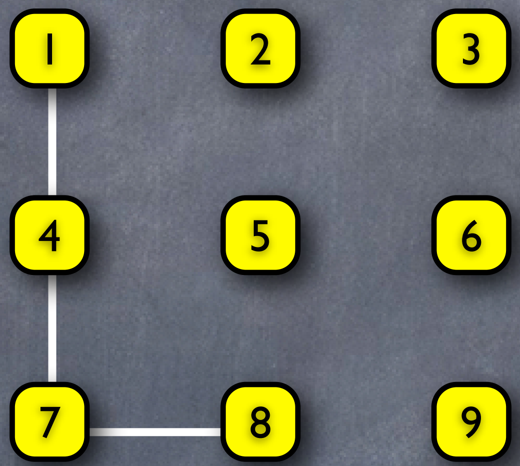

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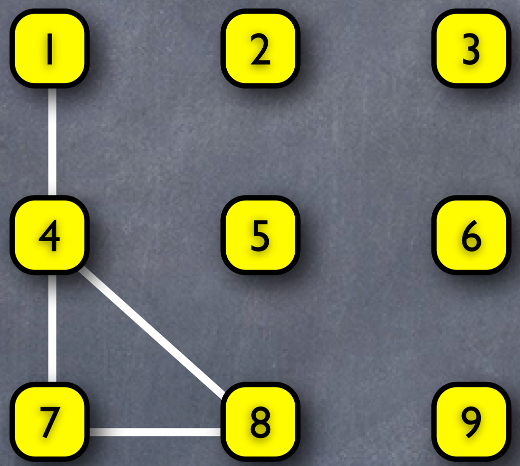

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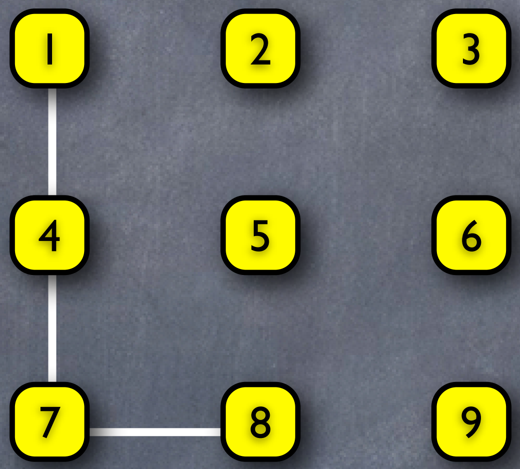

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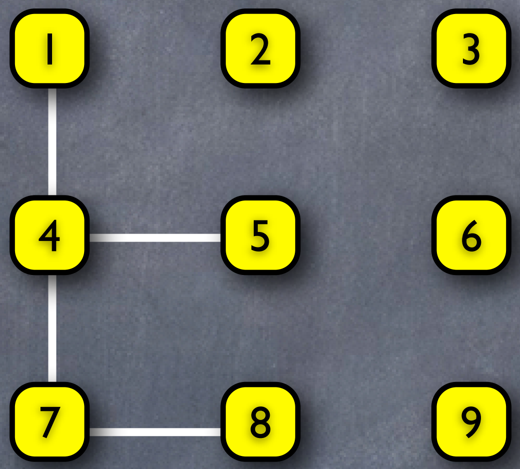

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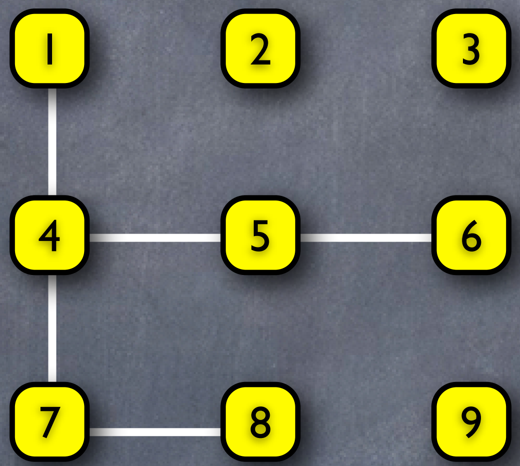

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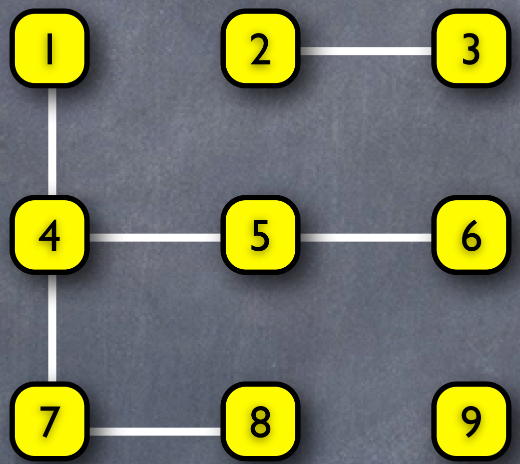

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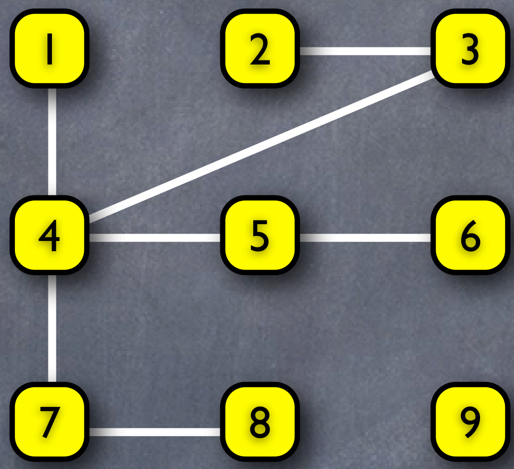

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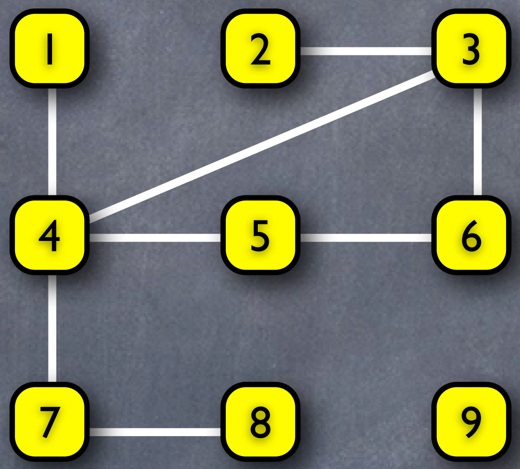

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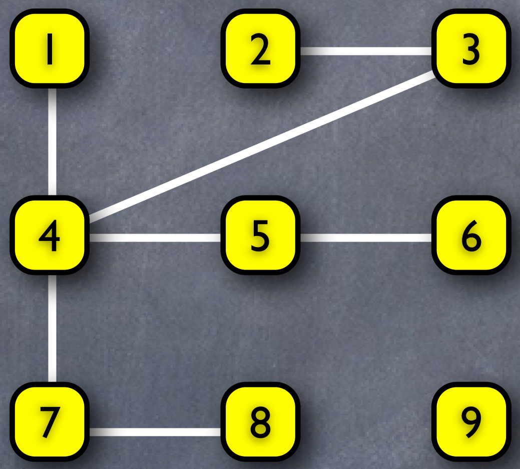

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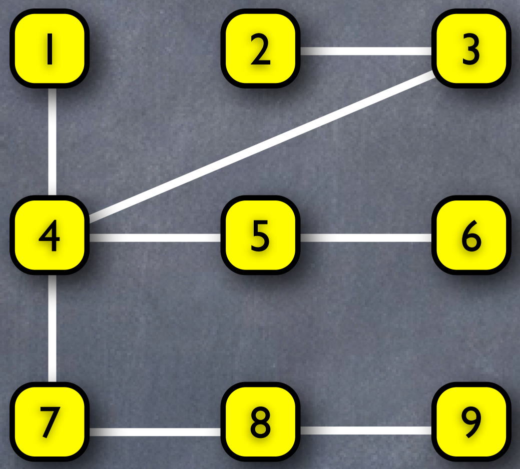

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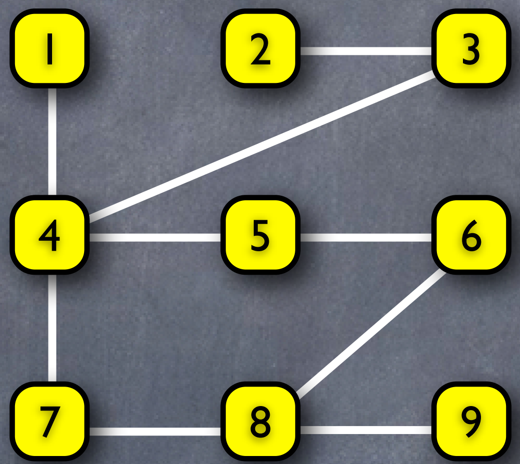

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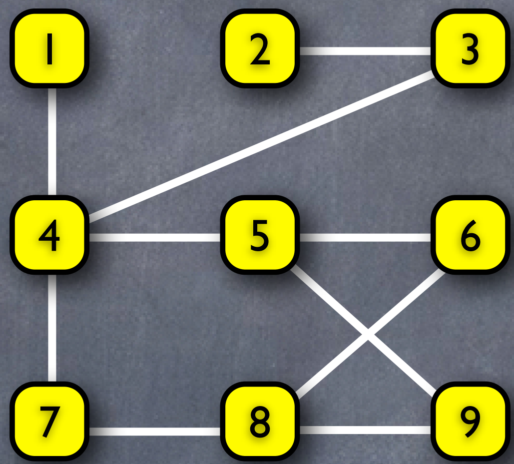

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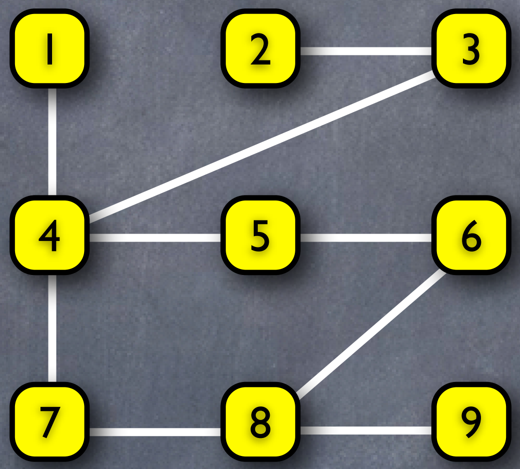

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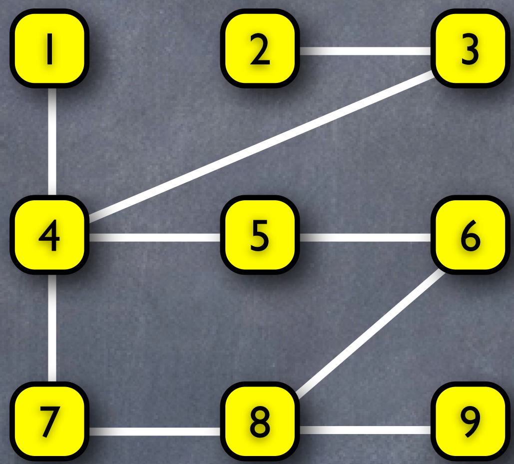

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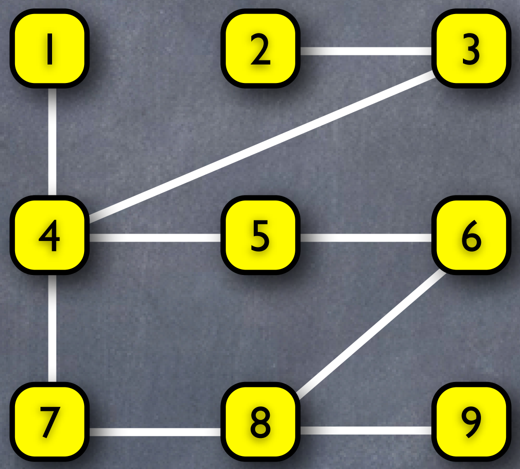

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- Lemma: $O(n^{3/2})$ edges stored since shortest cycle among stored edges has length at least 5.

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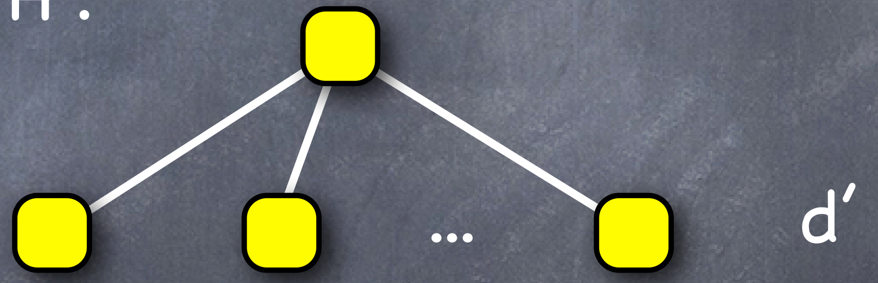

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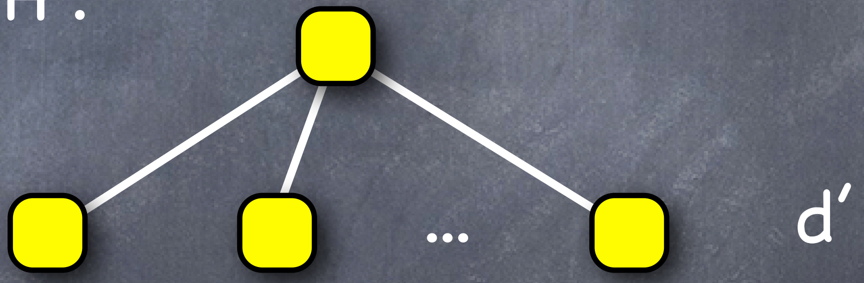

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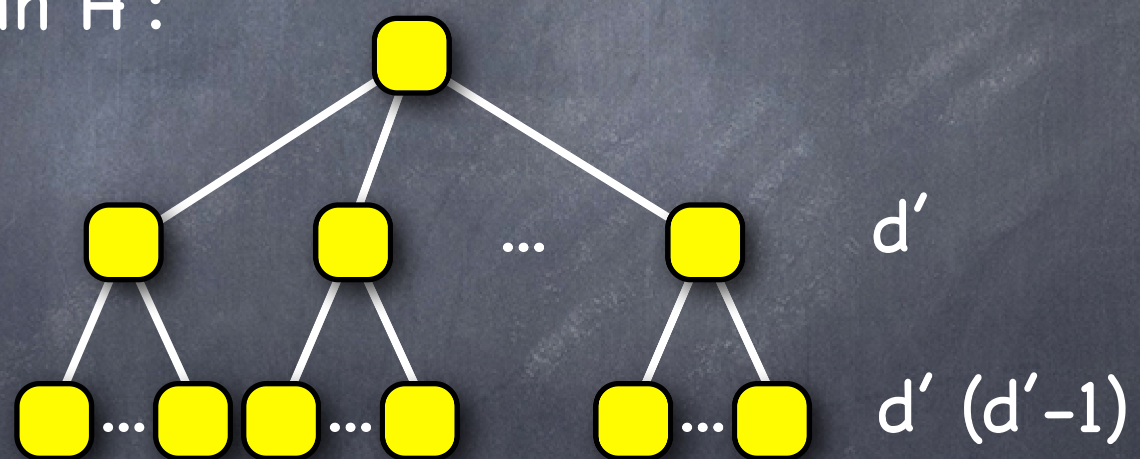

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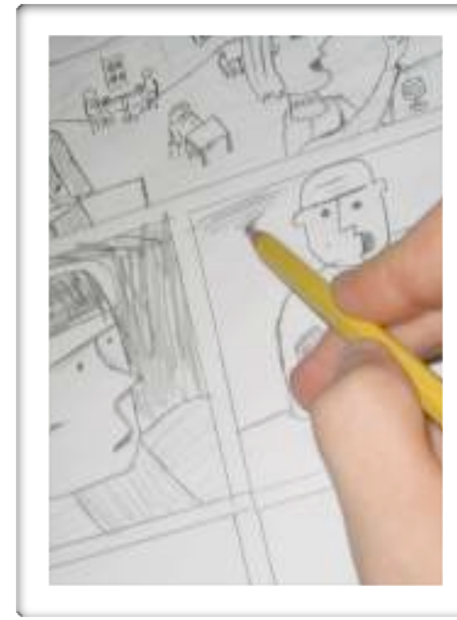
Spanners Summary

- *Thm:* There's a $O(n^{1+1/t})$-space stream algorithm returns a (2t-1)-spanner. [Feigenbaum, Kannan, McGregor, Suri, Zhang 05]
- *Extension:* Can process weighted graphs by rounding weights and constructing spanners for each weight class.

I. **Spanners**

II. **Sparsifiers**



III. **Sketches**

II. **Sparsifiers**

Synopsis for Cut Estimation
Merge-Reduce Stream Algorithm
Extensions

Sparsifiers & Cuts

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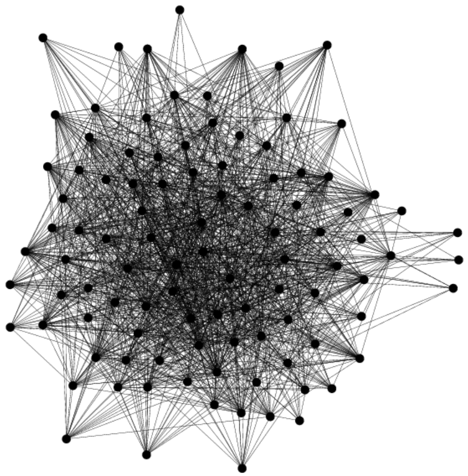
- *Measure:* Given a cut (L,R), the size of a cut $c_G(L,R)$ is the weight of all edges crossing the cut.



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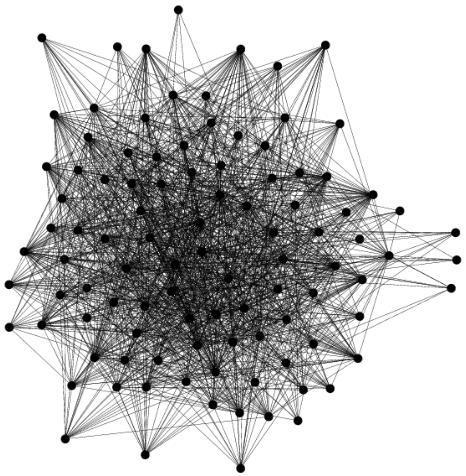

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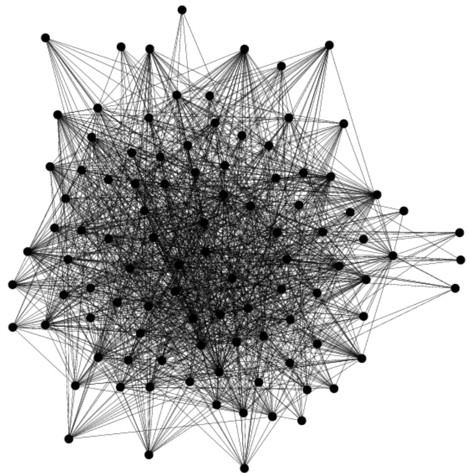

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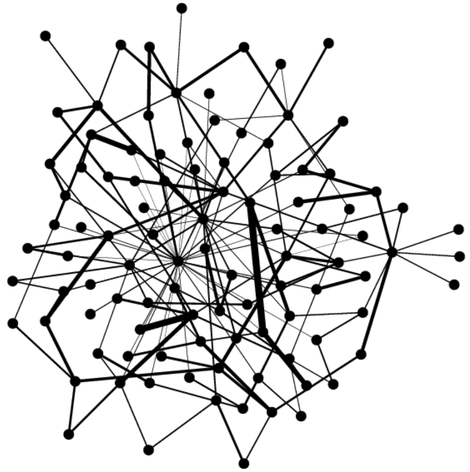

Sparsifier Graph H

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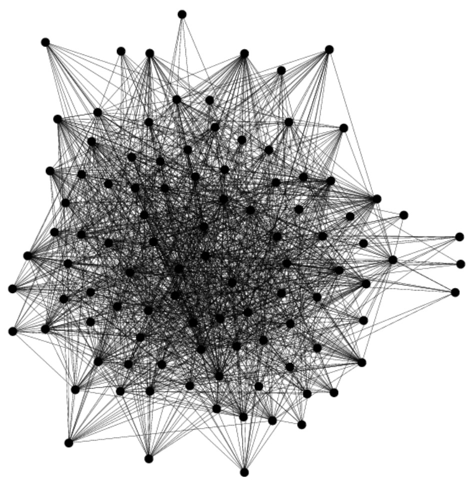

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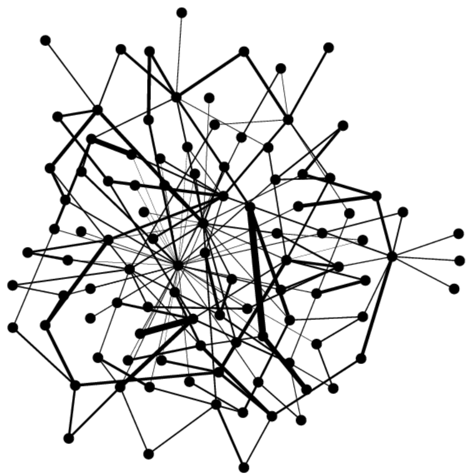

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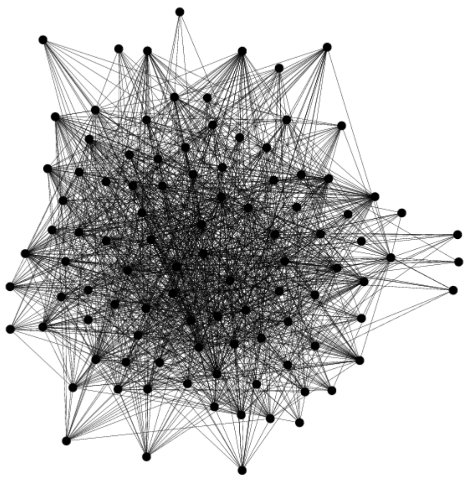
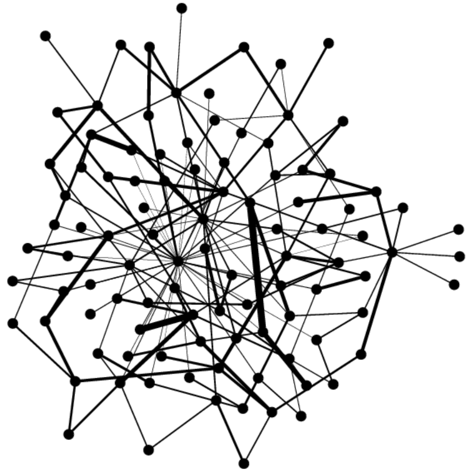
- *Thm (Benzur-Karger):* For any graph G there exists a (1+ε) sparsifier with only $O(\varepsilon^{-2} n)$ edges.

Sparsifiers & Cuts

- *Measure:* Given a cut (L,R), the size of a cut $c_G(L,R)$ is the weight of all edges crossing the cut.



Original Graph G



Sparsifier Graph H

Graphs courtesy of Nick Harvey

- *Synopsis:* A subgraph H of G is a $(1+\varepsilon)$ *sparsifier* if

$$c_G(L,R) \leq c_H(L,R) \leq (1+\varepsilon)\, c_G(L,R) \quad \text{for all cuts.}$$

- *Thm (Benzur-Karger):* For any graph G there exists a $(1+\varepsilon)$ sparsifier with only $O(\varepsilon^{-2} n)$ edges.
- *Thm:* Streaming construction in $O(\varepsilon^{-2} n \log^3 n)$ space.

Sparsifier: Algorithm

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- Main Idea: Segment stream as E_1, E_2, … each of size $O(\varepsilon^{-2}n)$. Let H_1 be $(1+\gamma)$ sparsifier of $E_1 \cup E_2$ etc.

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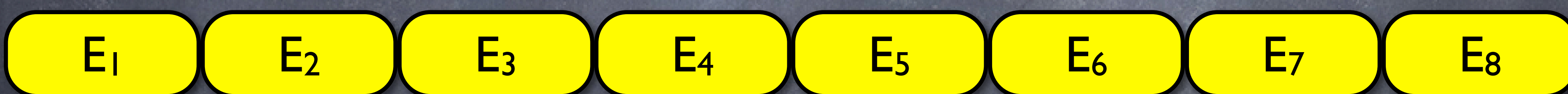

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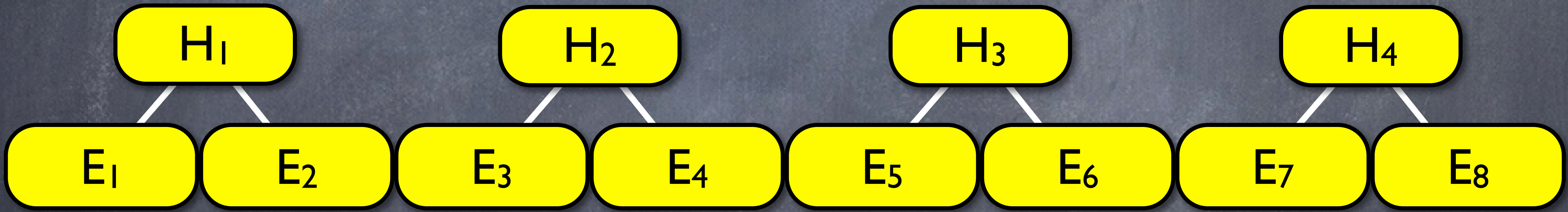

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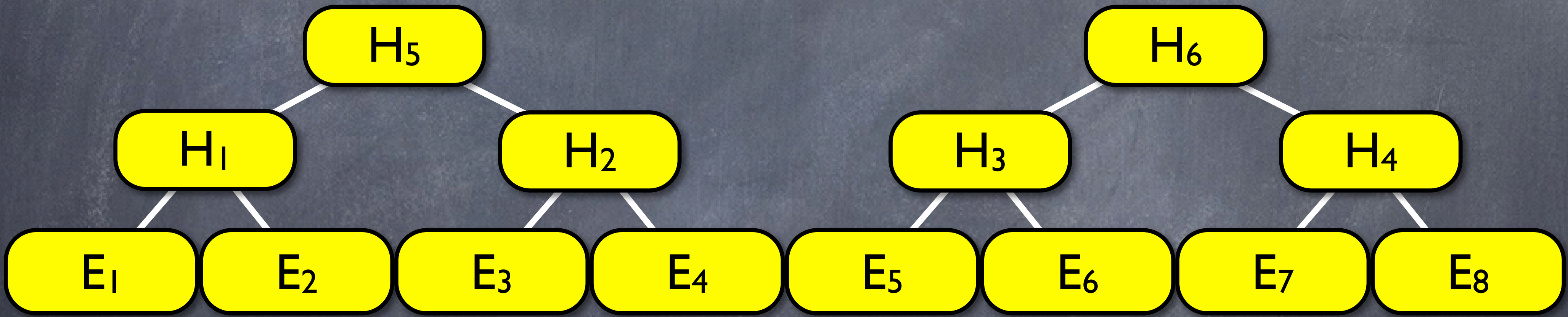

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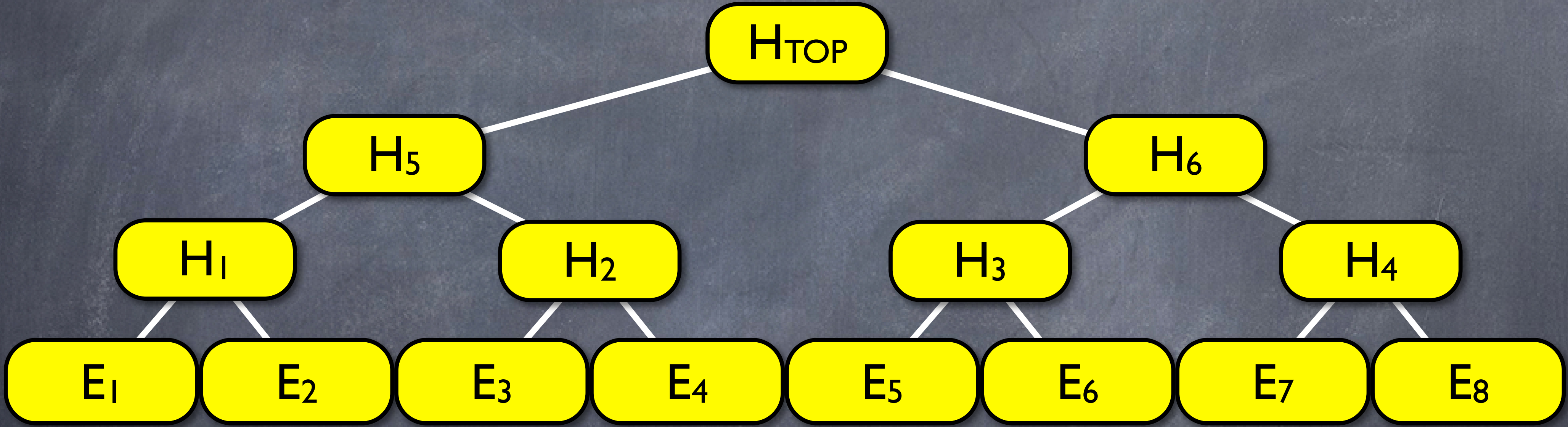

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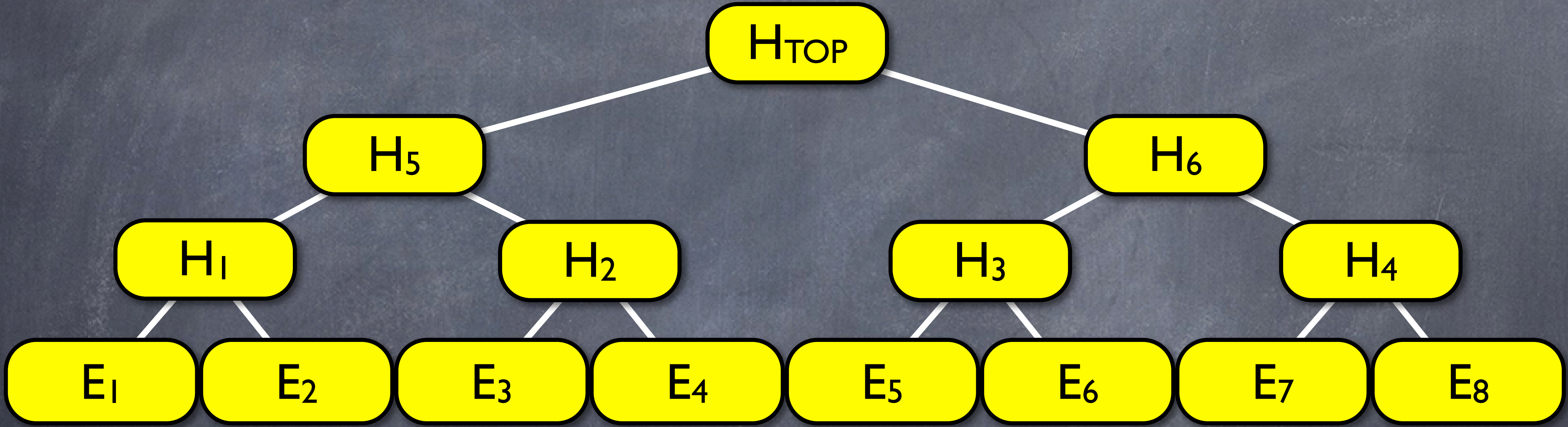

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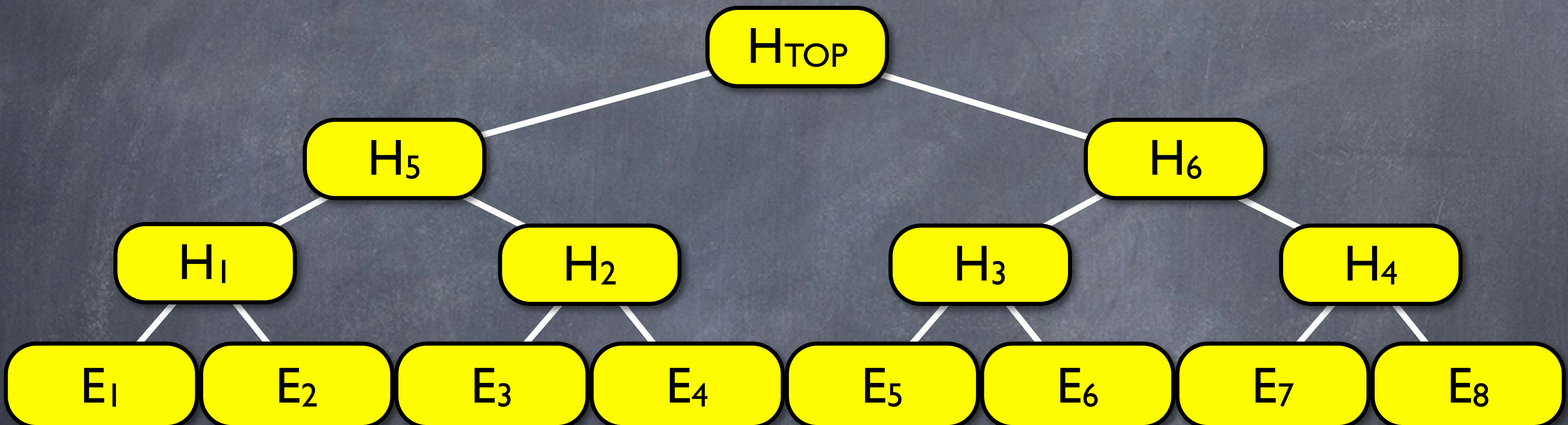

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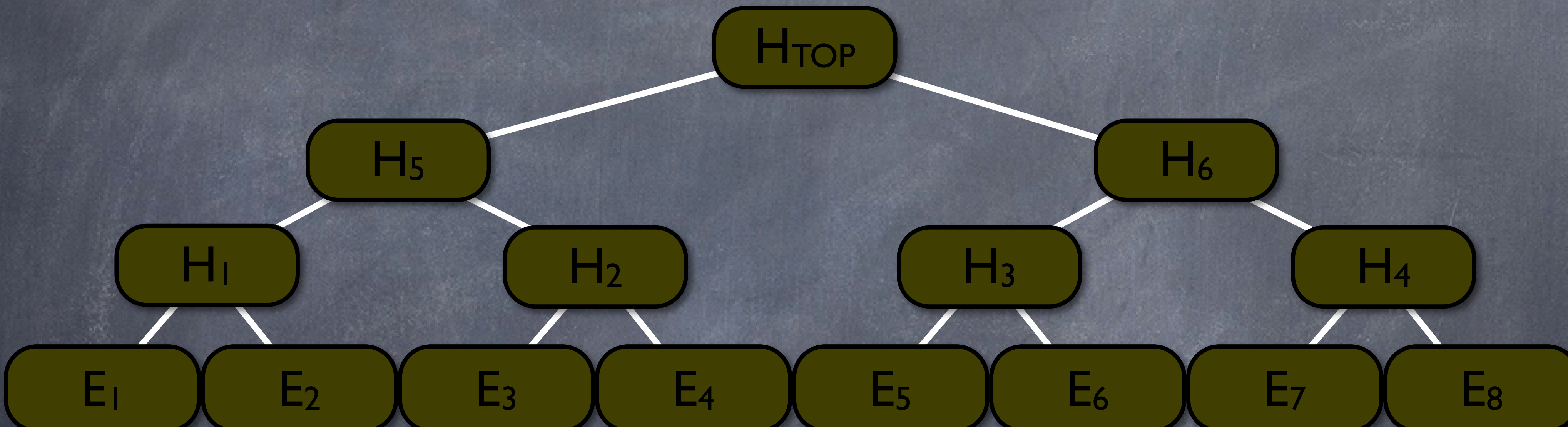

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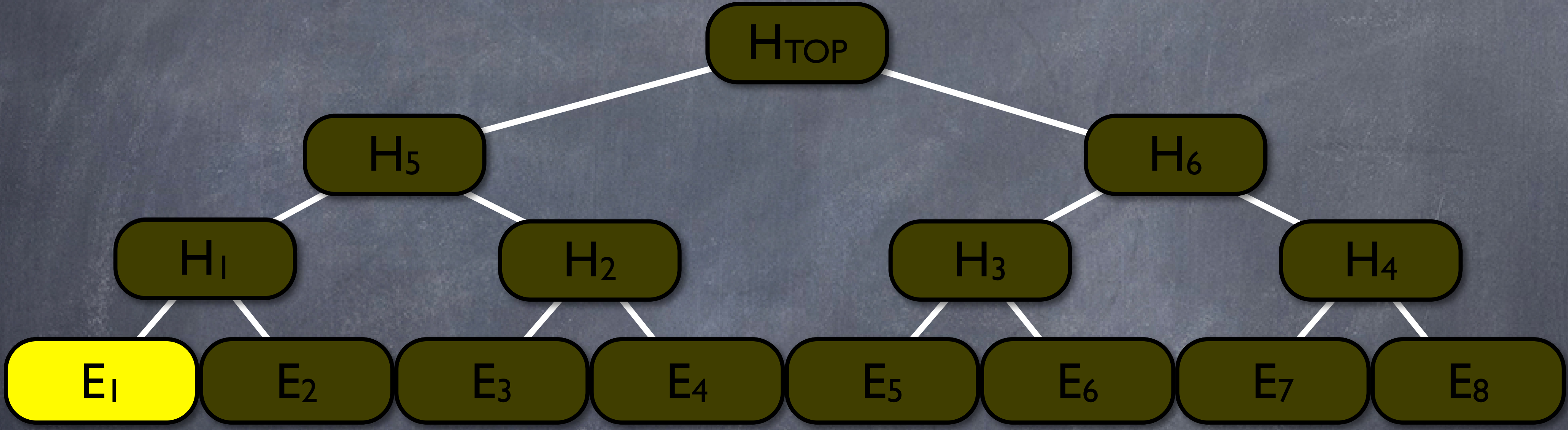

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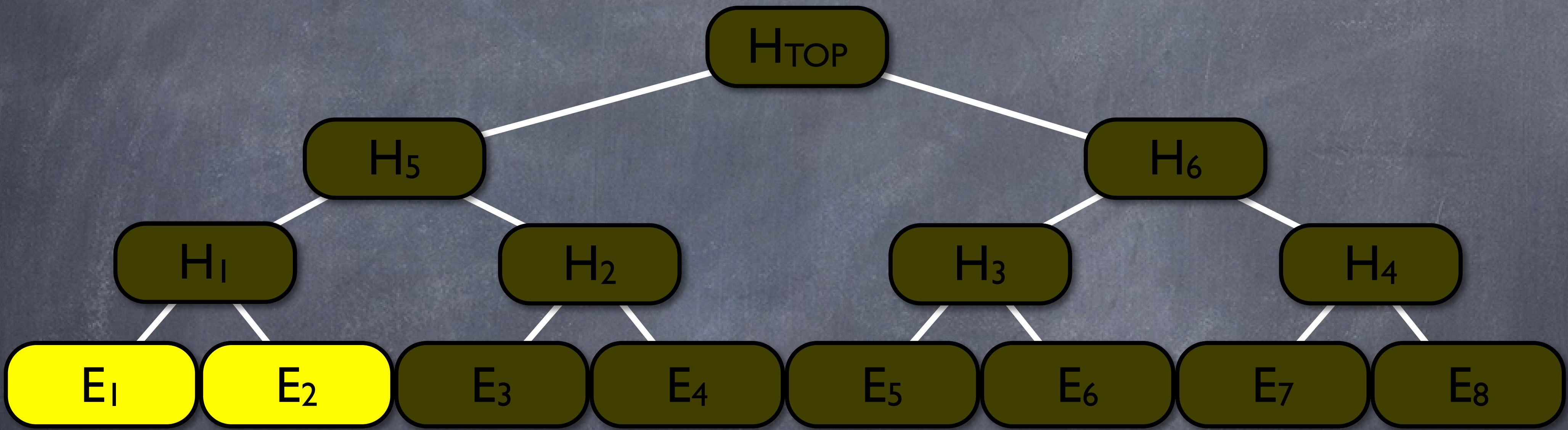

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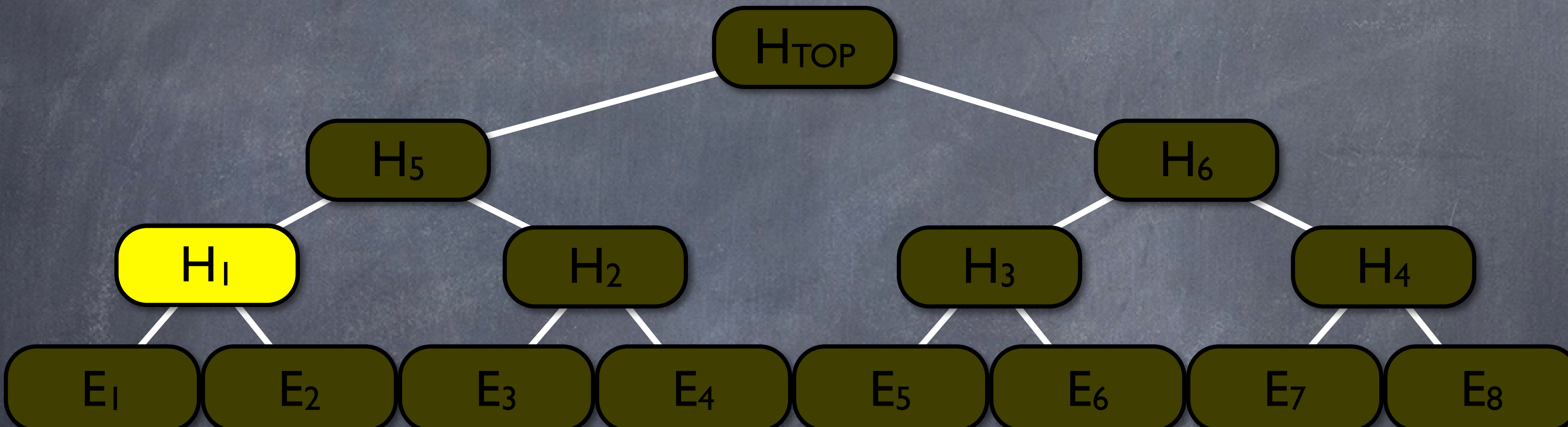

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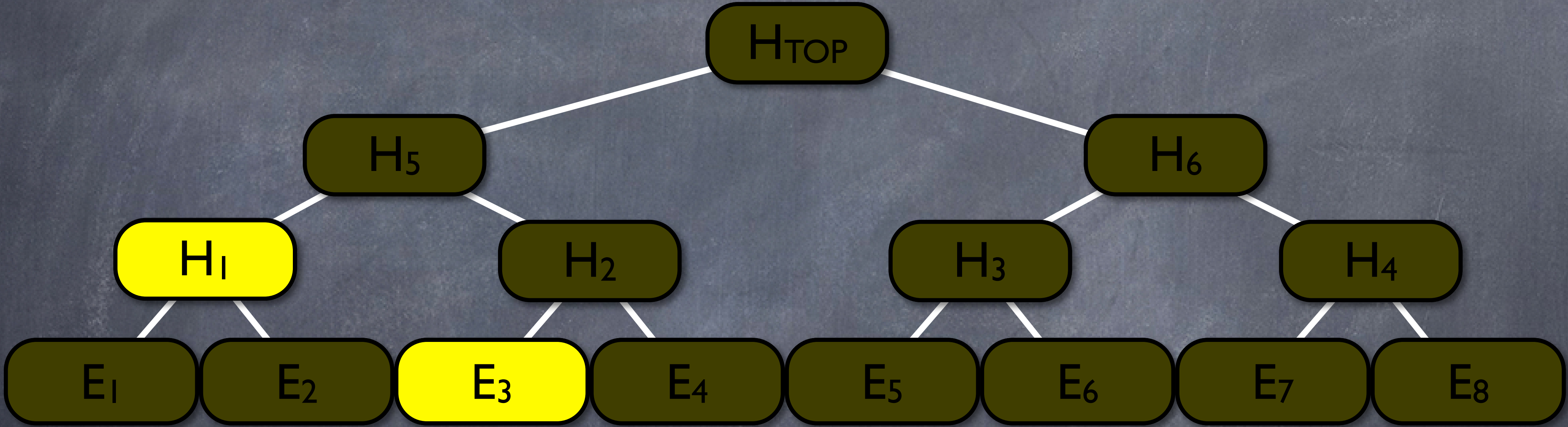

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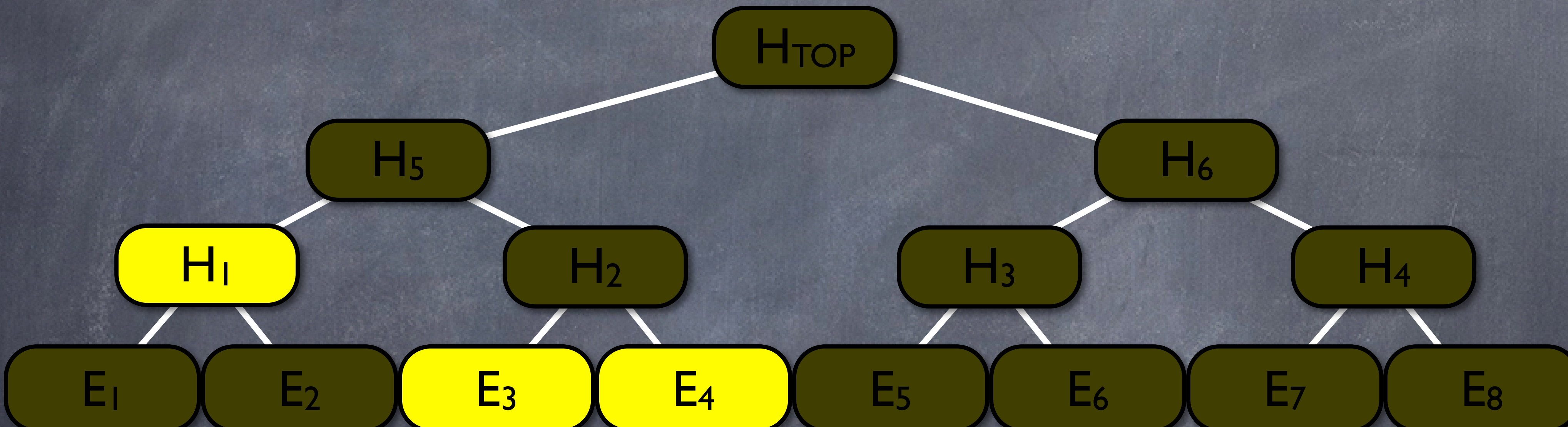

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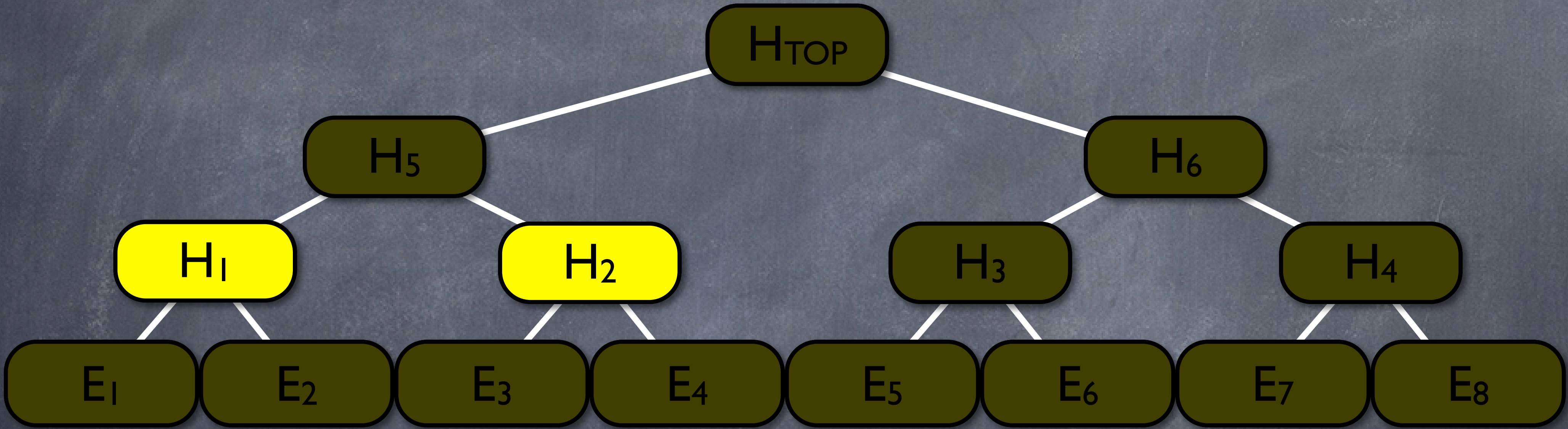

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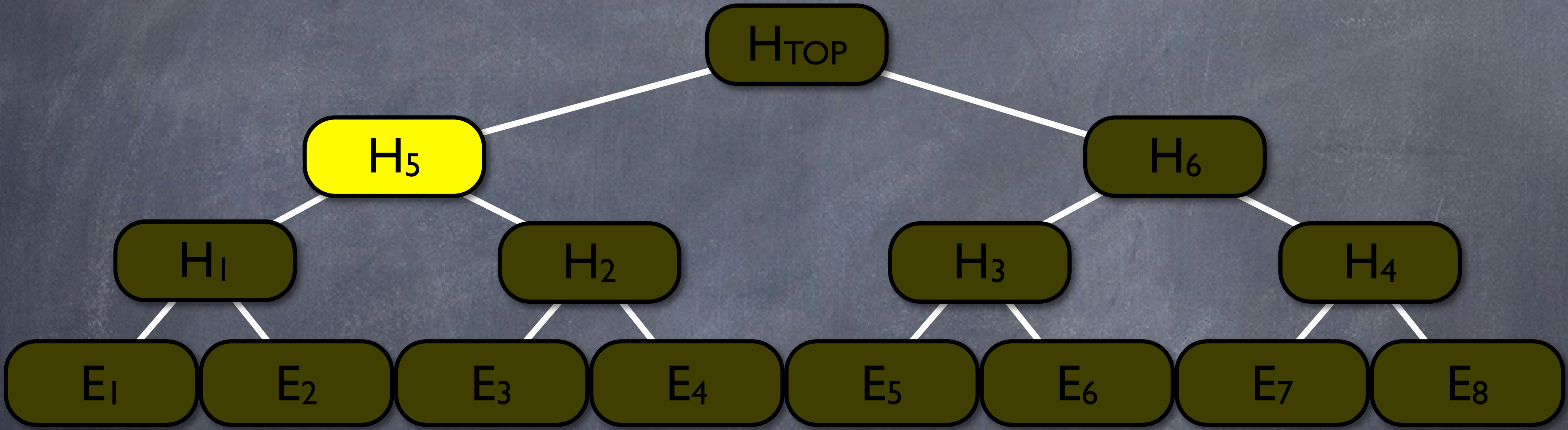

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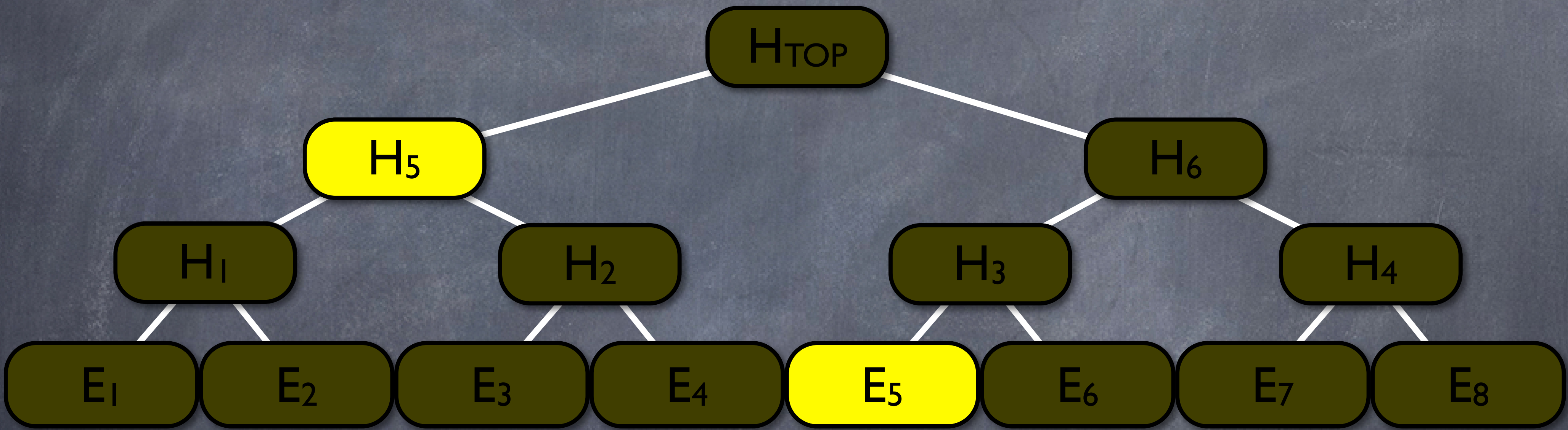

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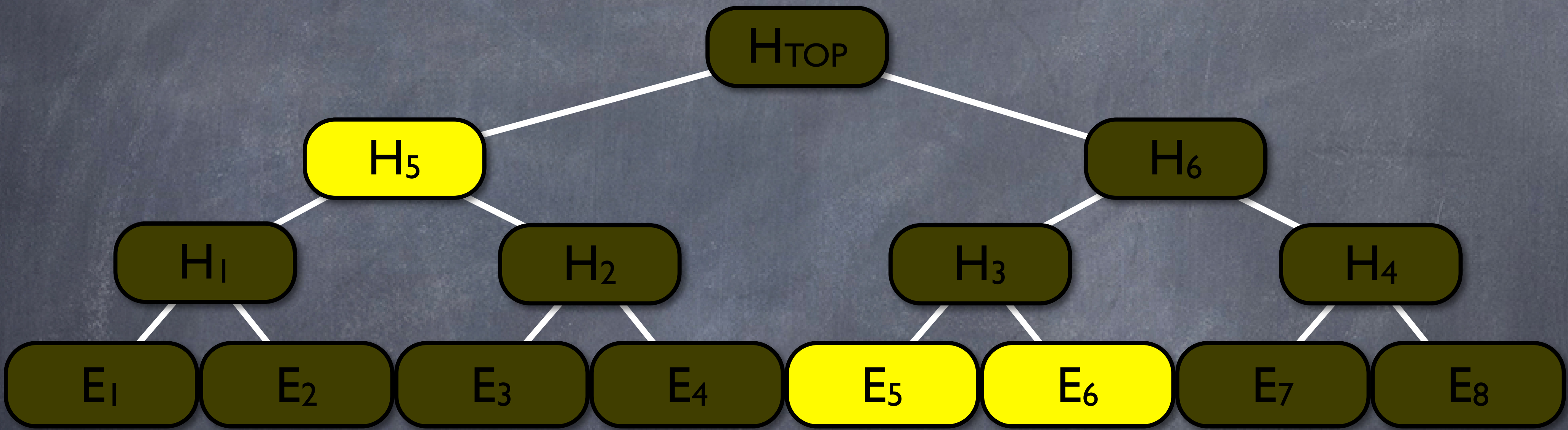

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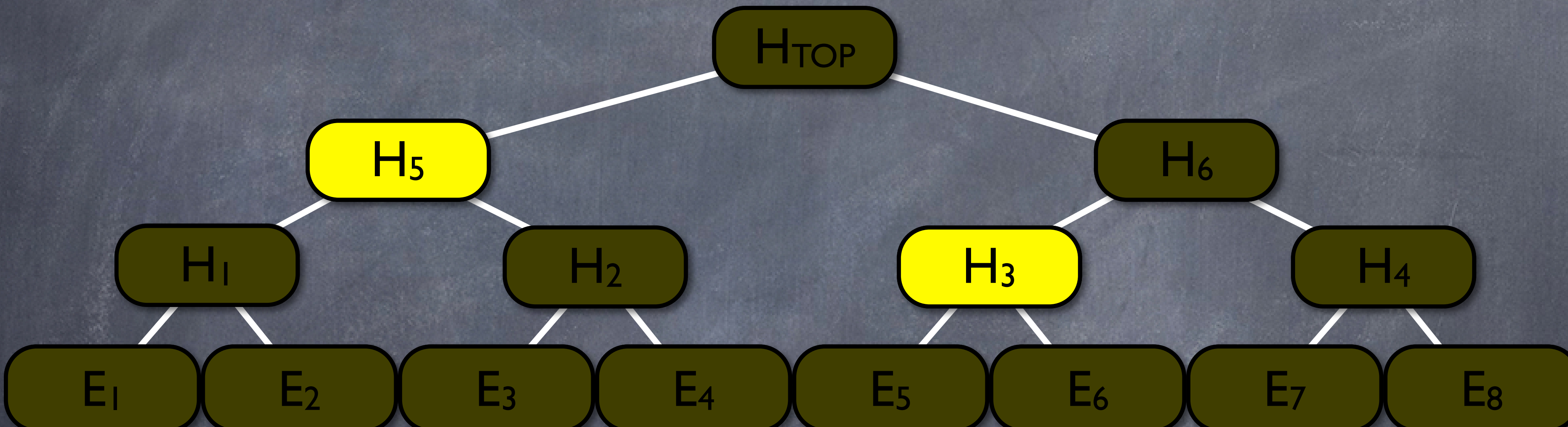

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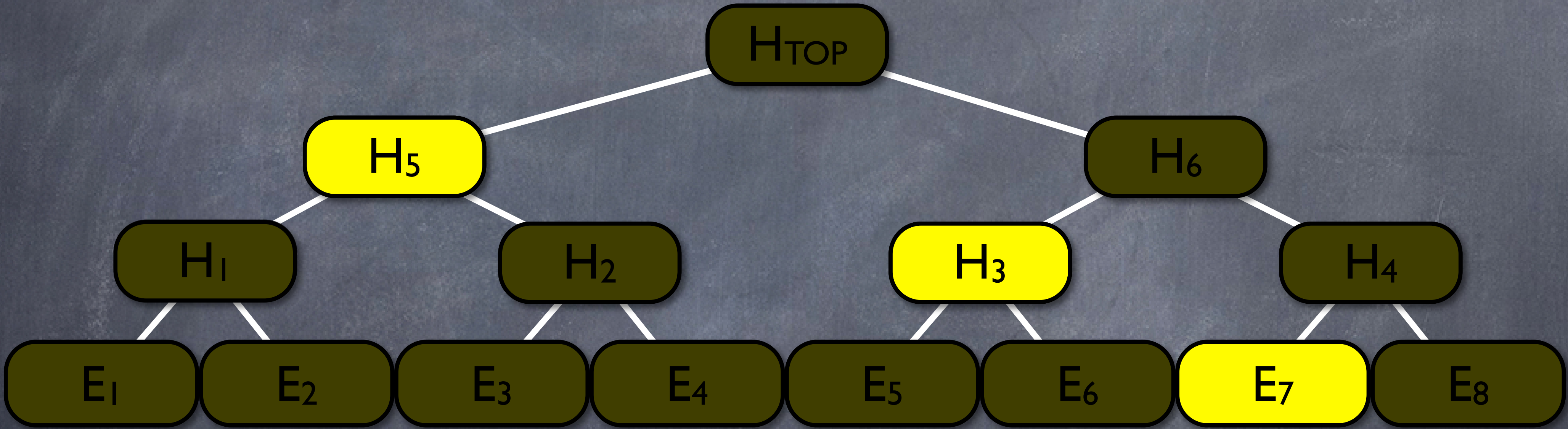

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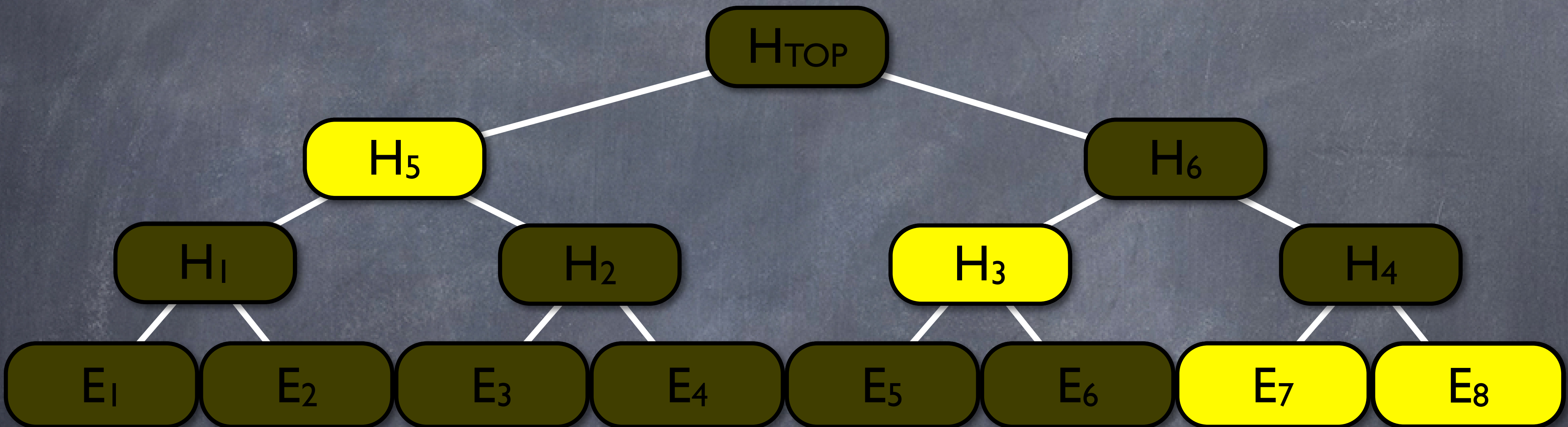

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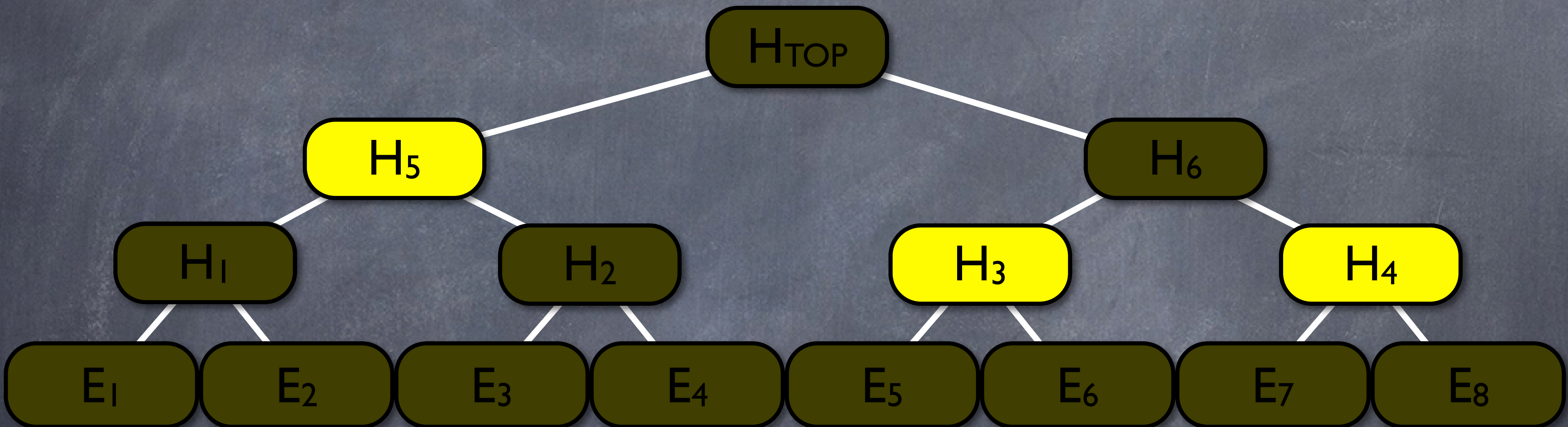

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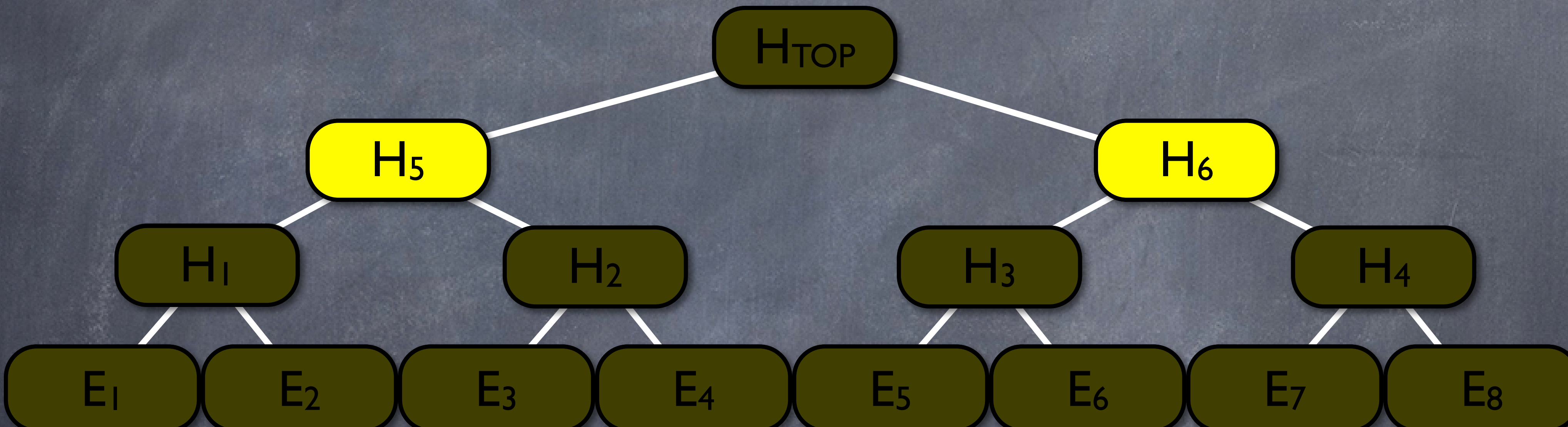

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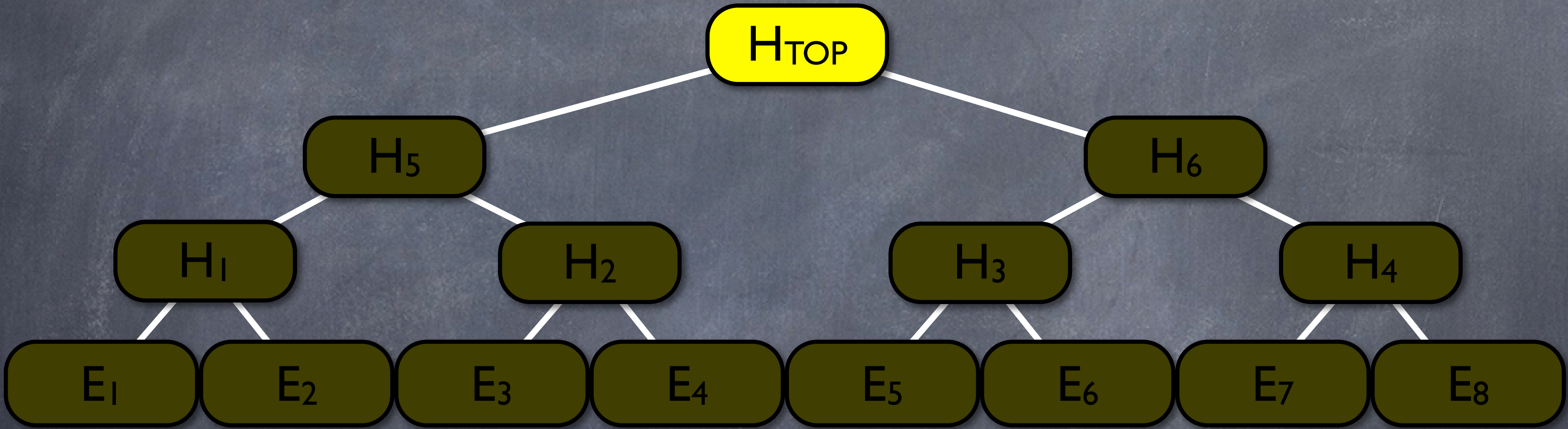

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Sparsifier Summary

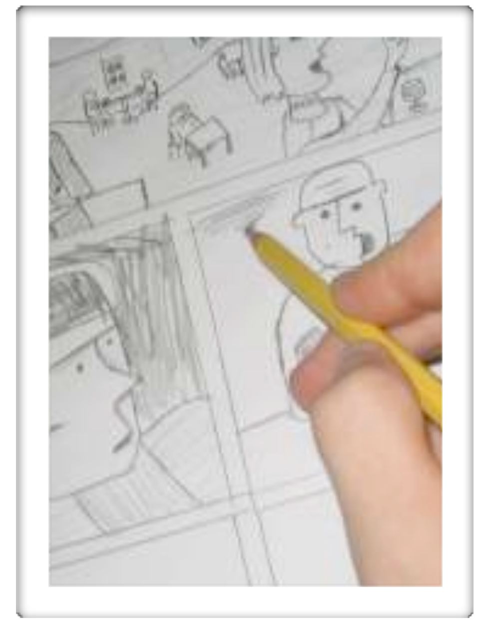
- *Thm:* A $(1+\varepsilon)$ sparsifier of a graph can be constructed in $O(\varepsilon^{-2}\, n \text{ polylog } n)$ space.

  [Ahn, Guha 09], [Goel, Kapralov, Khanna 10], [Sidiropoulos 10]

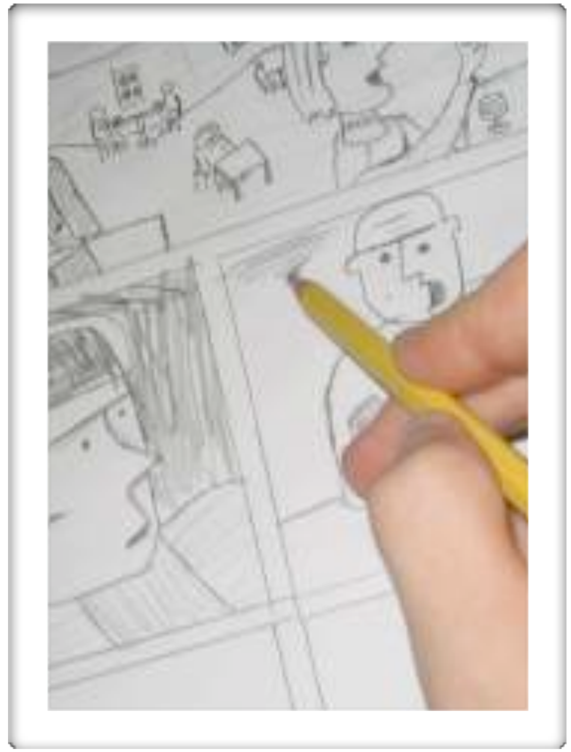
- Generalizes to spectral sparsification which preserves properties relating to random walks. [Kelner, Levin 11]

I. **Spanners**

II. **Sparsifiers**



III. **Sketches**



III. **Sketches**

Family of Linear Synopses
Distributed & Supports Deletions
Two Connectivity Examples

Linear Sketches

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$$\begin{bmatrix} \\ \\ v \\ \\ \\ \end{bmatrix}$$

Linear Sketches

- <u>*Random linear projection:*</u> M: $\mathbb{R}^n \rightarrow \mathbb{R}^k$ (where $k \ll n$) that preserves properties of any $v \in \mathbb{R}^n$ with high probability.

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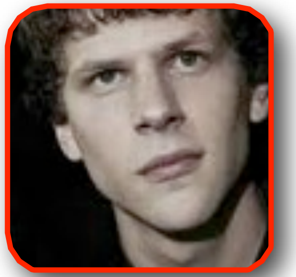
? *Question:* What about analyzing massive graphs via sketches?

Distributed Data

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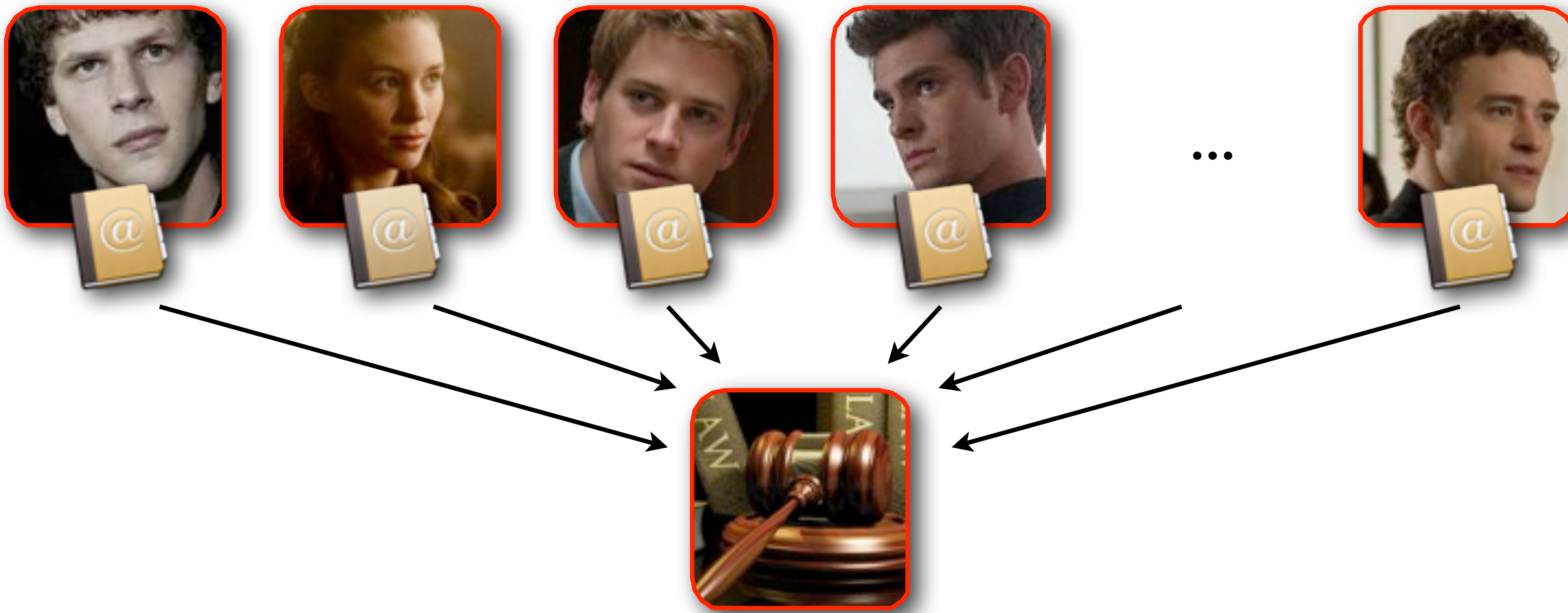

- ***Input:*** Each player knows neighborhood Γ(v) for a node v
- ***Goal:*** Simultaneously, each player sends O(polylog n) bits to a central player who then determines if graph is connected.

This can't be possible?!

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  c) Participants may have Ω(n) friends.

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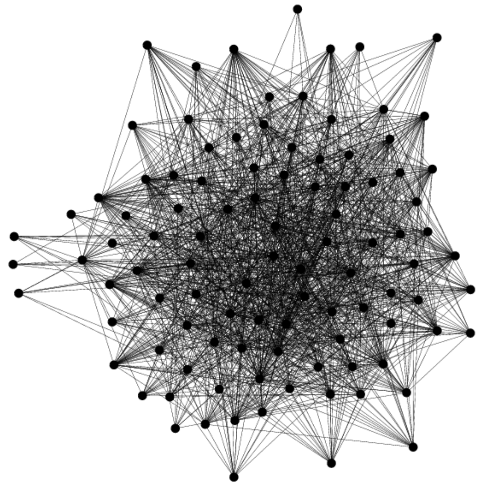

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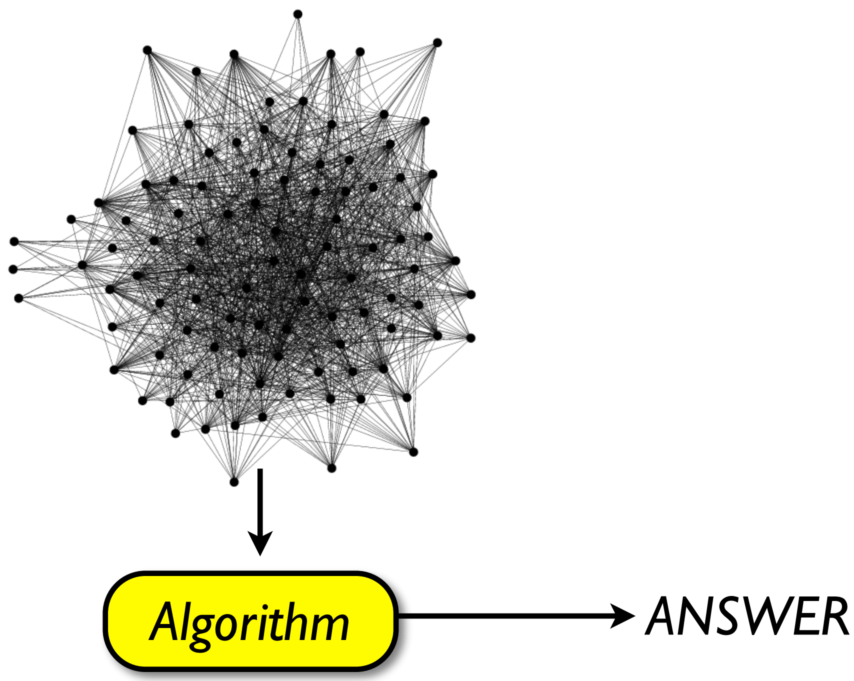

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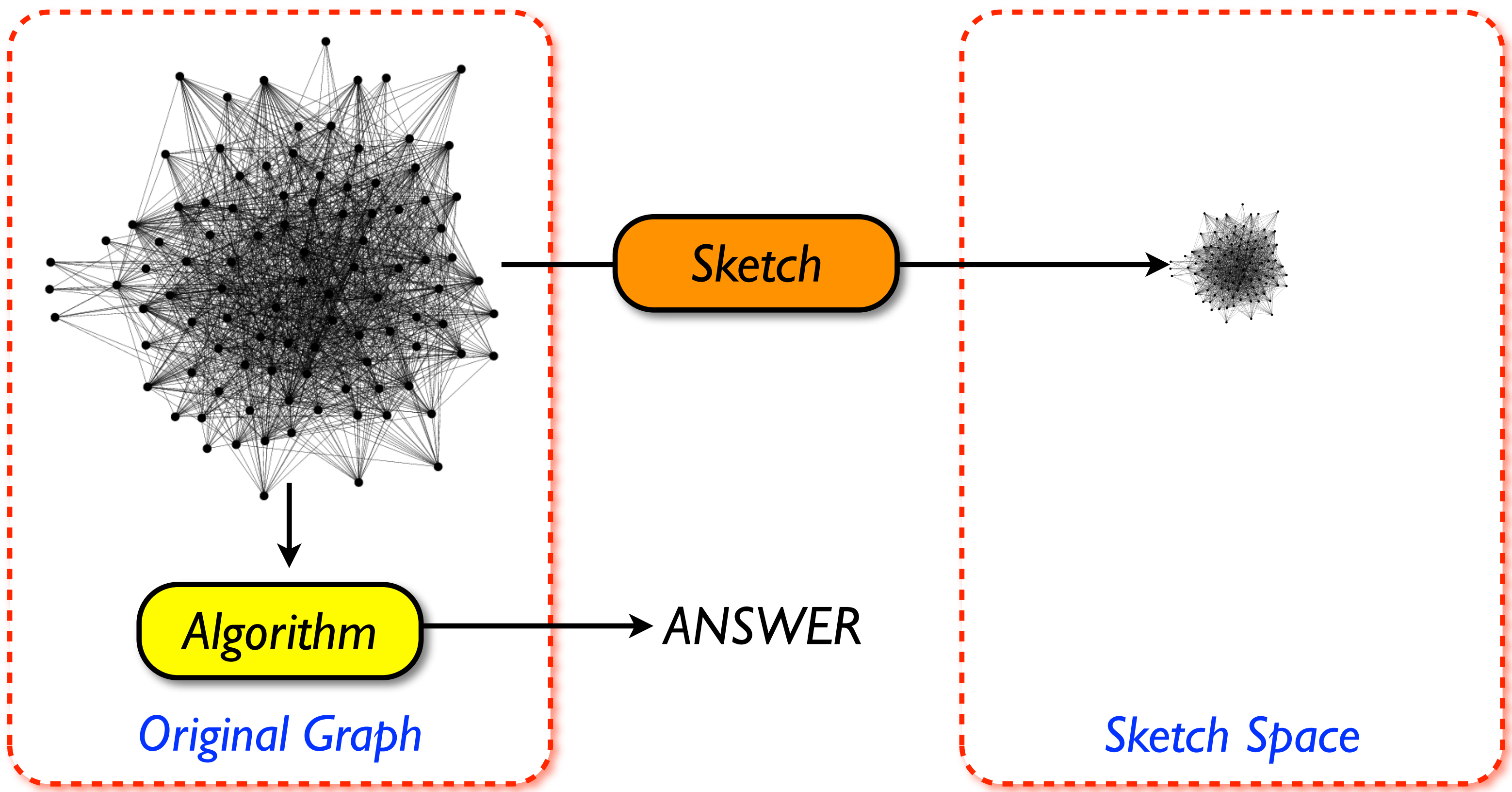

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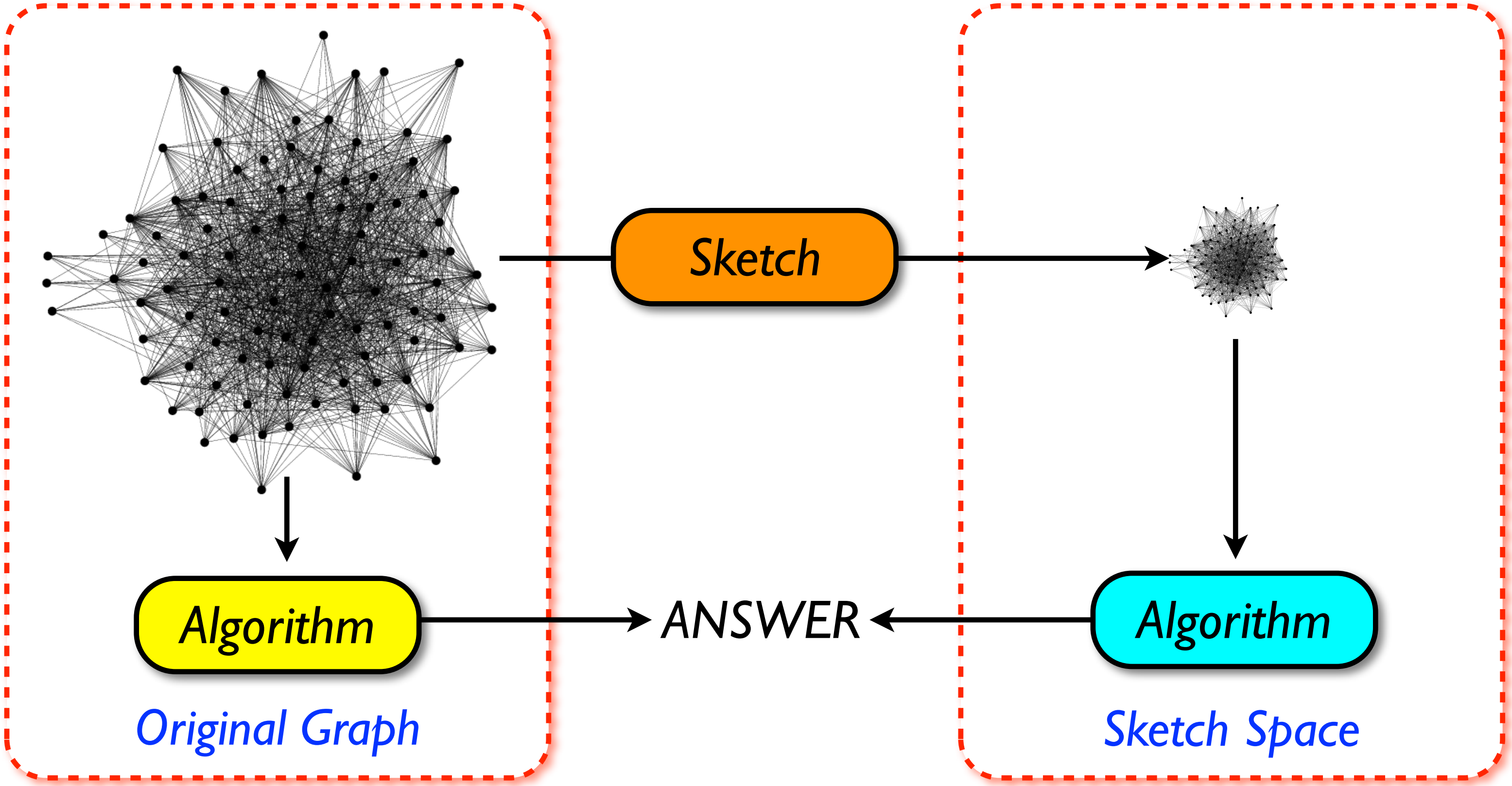

Two Examples

First Theorem: Testing Connectivity

a) *Dynamic Graph Stream:* O(n polylog n) space.

b) *Distributed Setting:* O(polylog n) length messages.

Second Theorem: Checking every cut has size $\geq$ k

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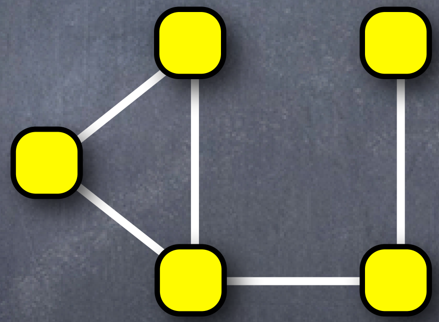

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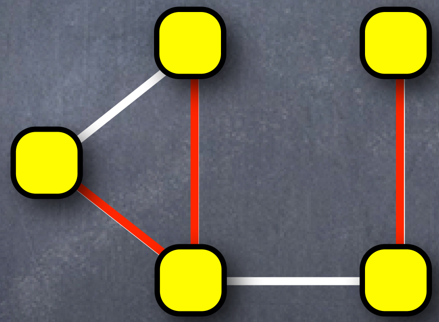

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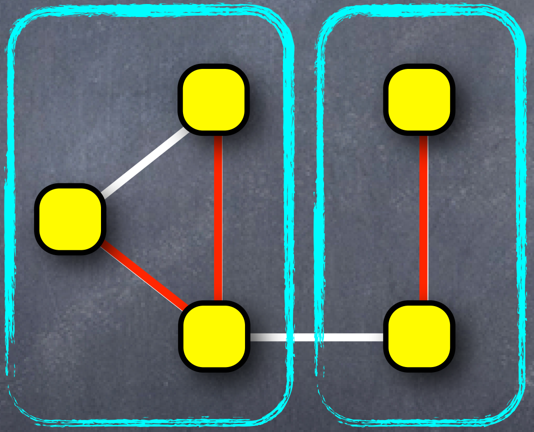

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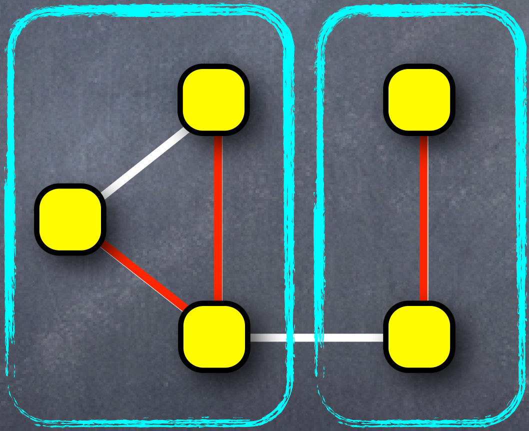

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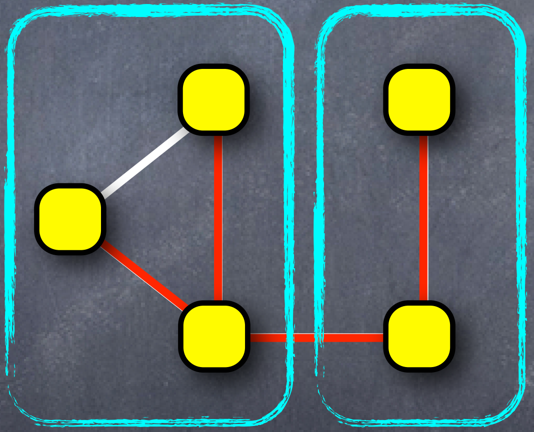

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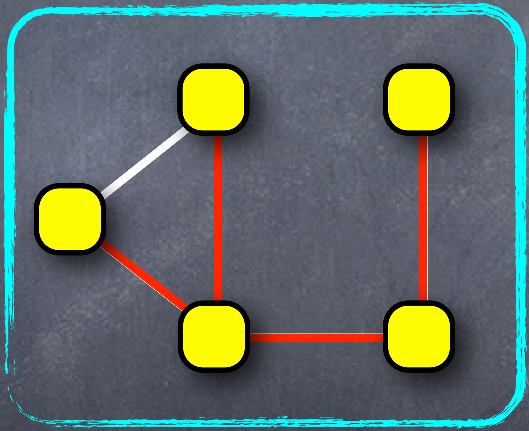

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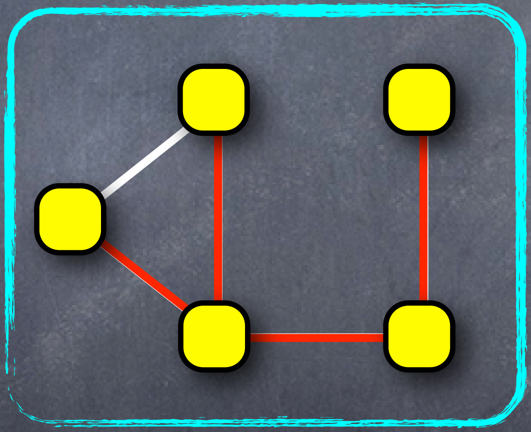

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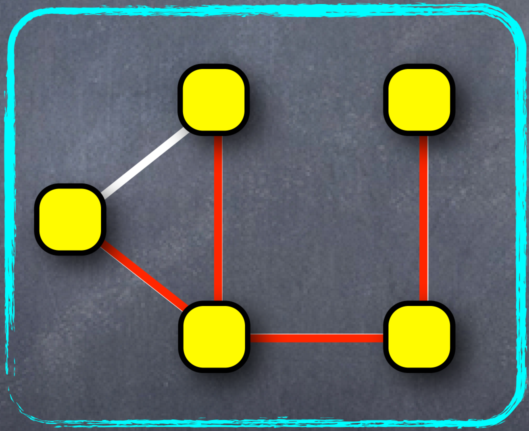

- Lemma: After O(log n) rounds selected edges include spanning forest.

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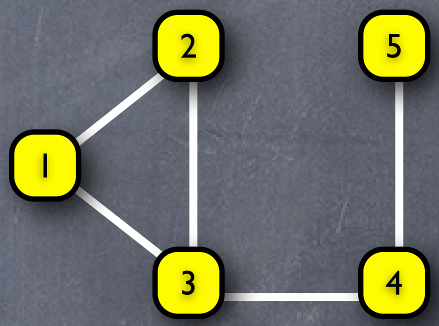

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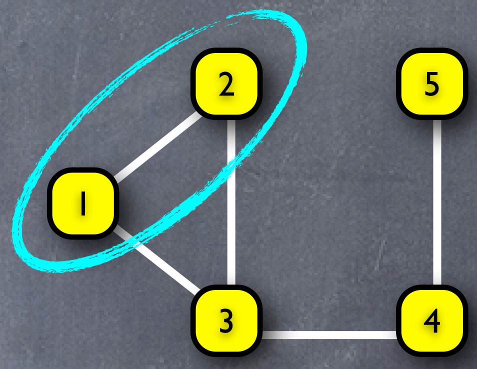

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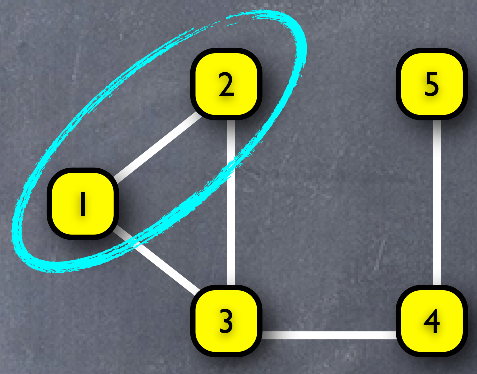

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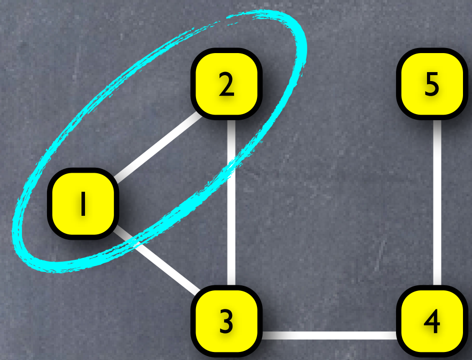

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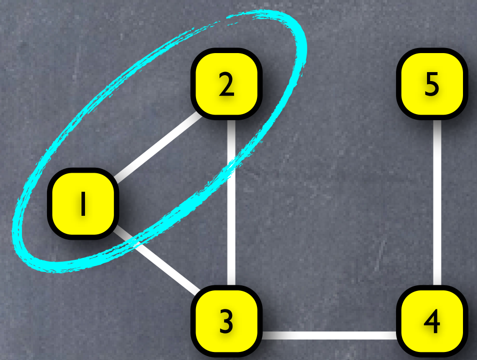

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- **Lemma:** $\exists$ random M: $\mathbb{R}^N \to \mathbb{R}^k$ with k=O(polylog N) such that for any $a \in \mathbb{R}^N$, with high probability

$$M\mathbf{a} \longrightarrow e \in \text{support}(\mathbf{a})$$

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Detail: Actually each player sends log n indept sketches M_1a_j, M_2a_j, … and central player uses M_ia_j when emulating i^{th} iteration of the algorithm.

Two Examples

First Theorem: Testing Connectivity

a) *Dynamic Graph Stream:* O(n polylog n) space.

b) *Distributed Setting:* O(polylog n) length messages.

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- Lemma: $G(V,F_1+\ldots+F_k)$ is k-connected iff $G(V,E)$ is.

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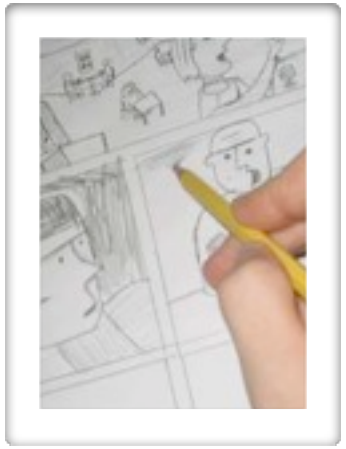
  - etc.

Sketches Summary

- *Graph Sketches:* Linear projections that preserve structural graph properties. Results *parallelizable*, *streamable*, and *support deletions*.
- *Talk Results:* Projecting O(n)-dimensional neighborhoods to O(polylog n) dimensions while preserving connectivity and cuts.
- *Other Results:* Spanners, Bipartiteness, MST, Triangles, Matching, ...



And over to Part II…

Sağ olun!