Data Streams, Dyck Languages, and *Detecting Dubious Data Structures*

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Ranganath Kondapally *Dartmouth College*
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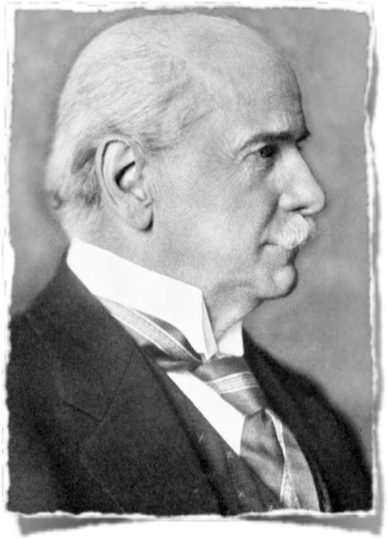

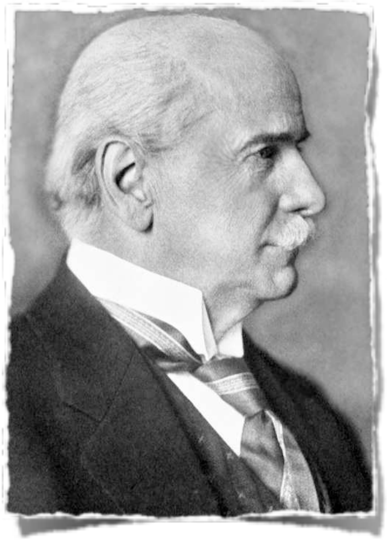

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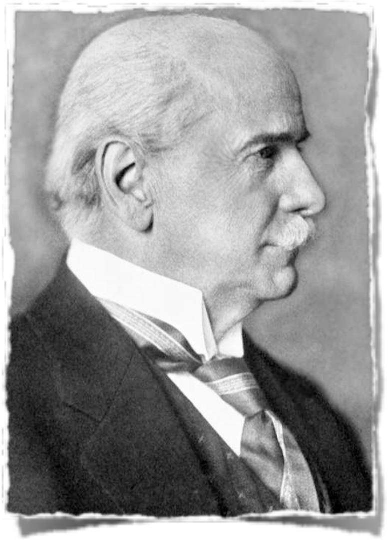

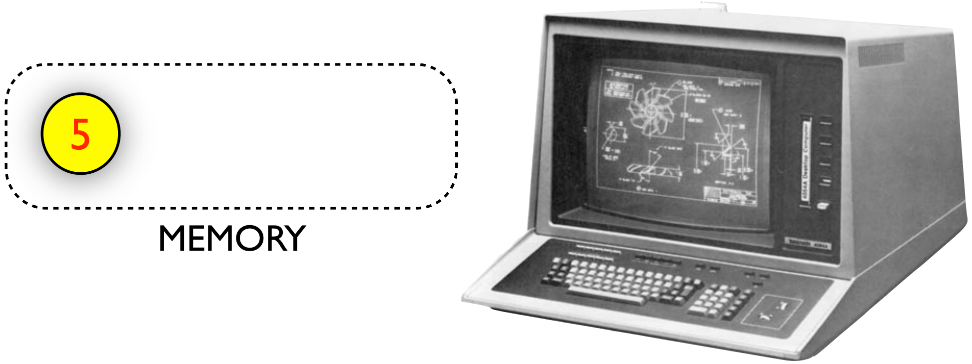

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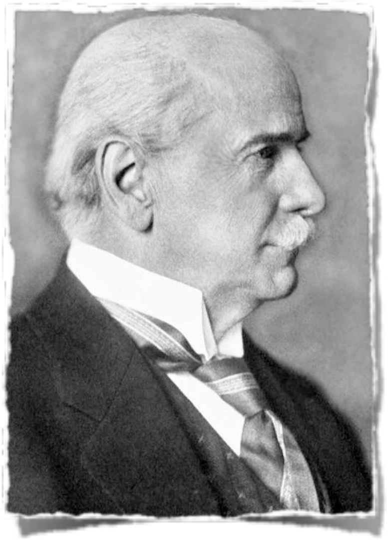
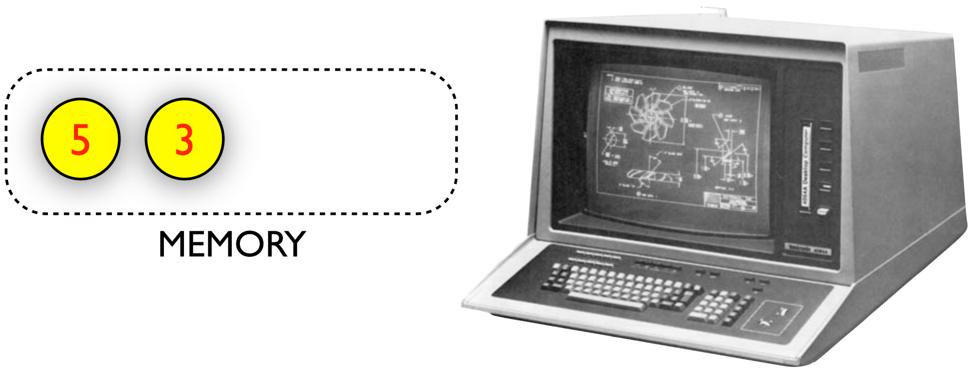
  ins(5)





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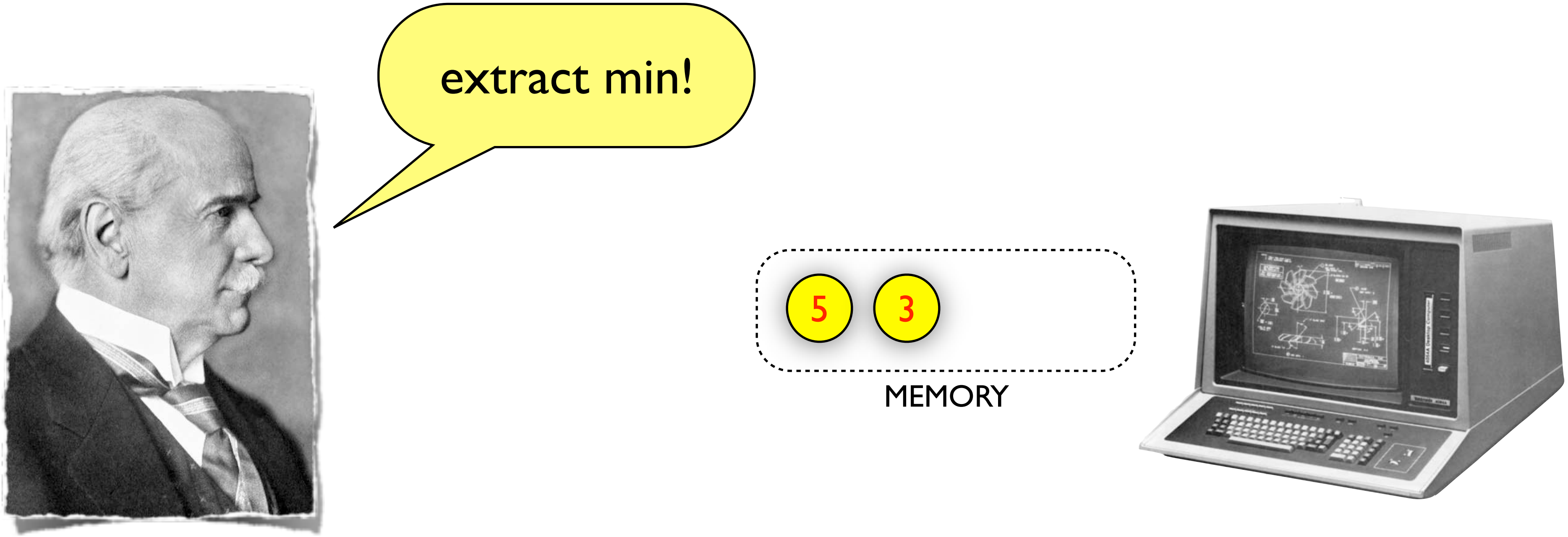

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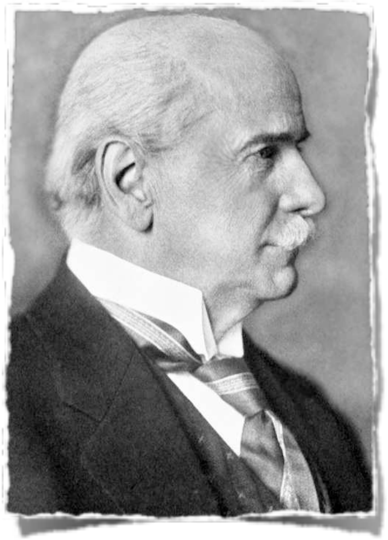

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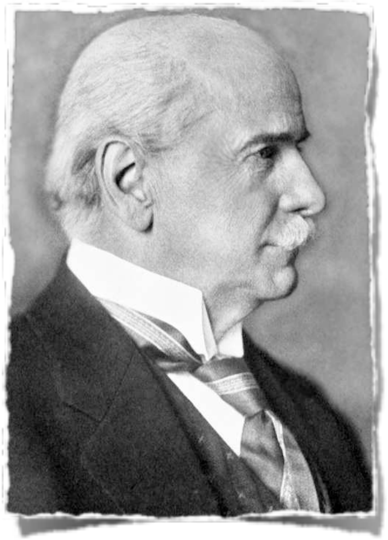

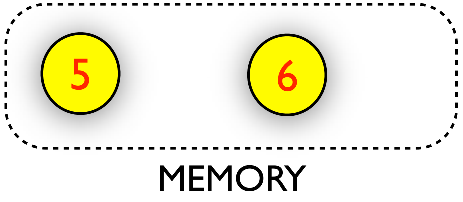

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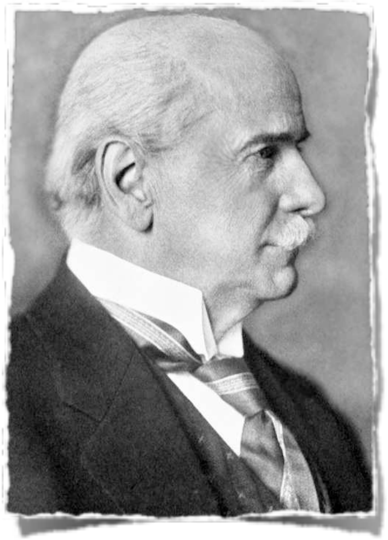

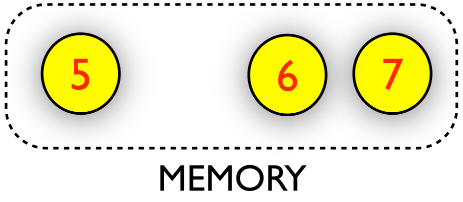

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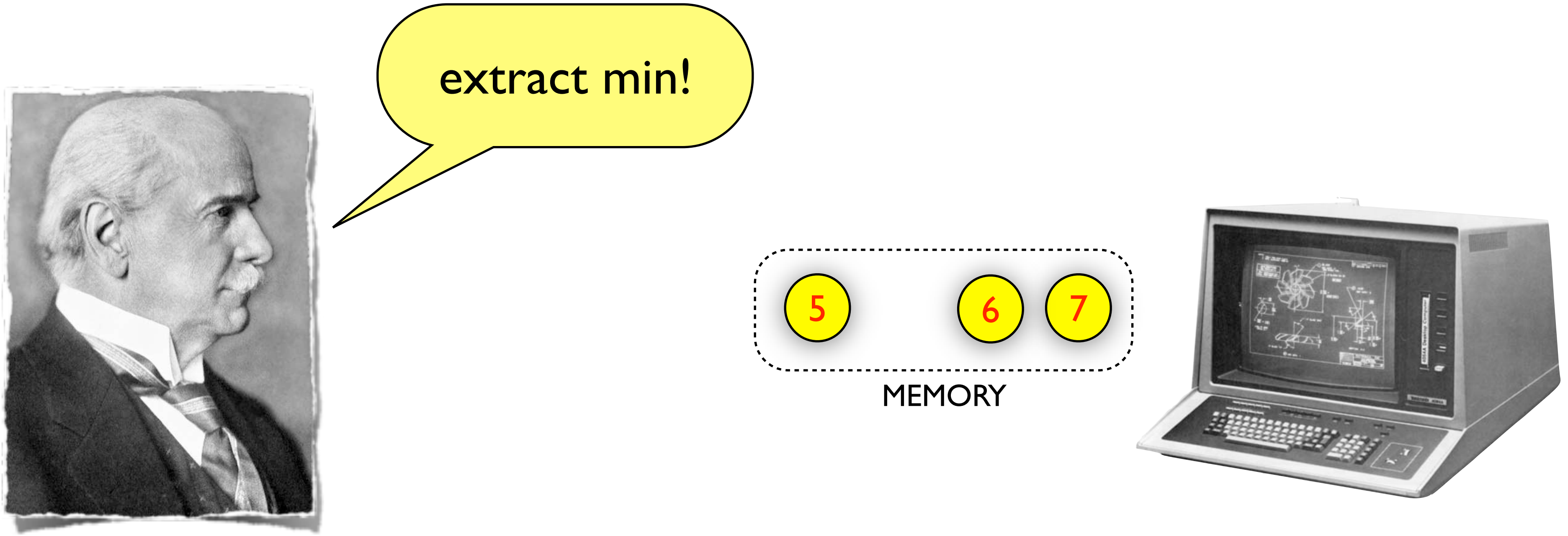

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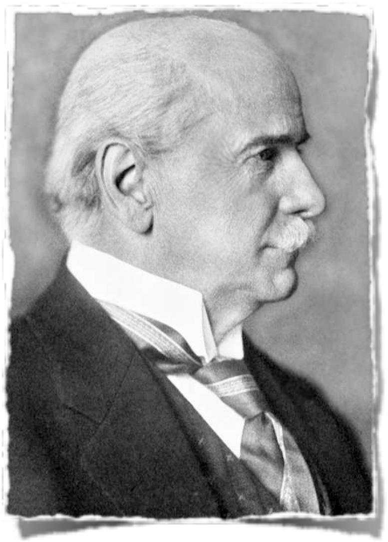

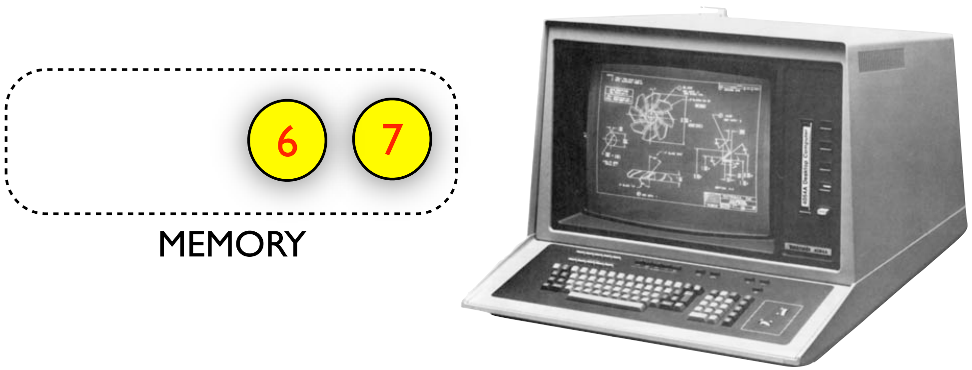

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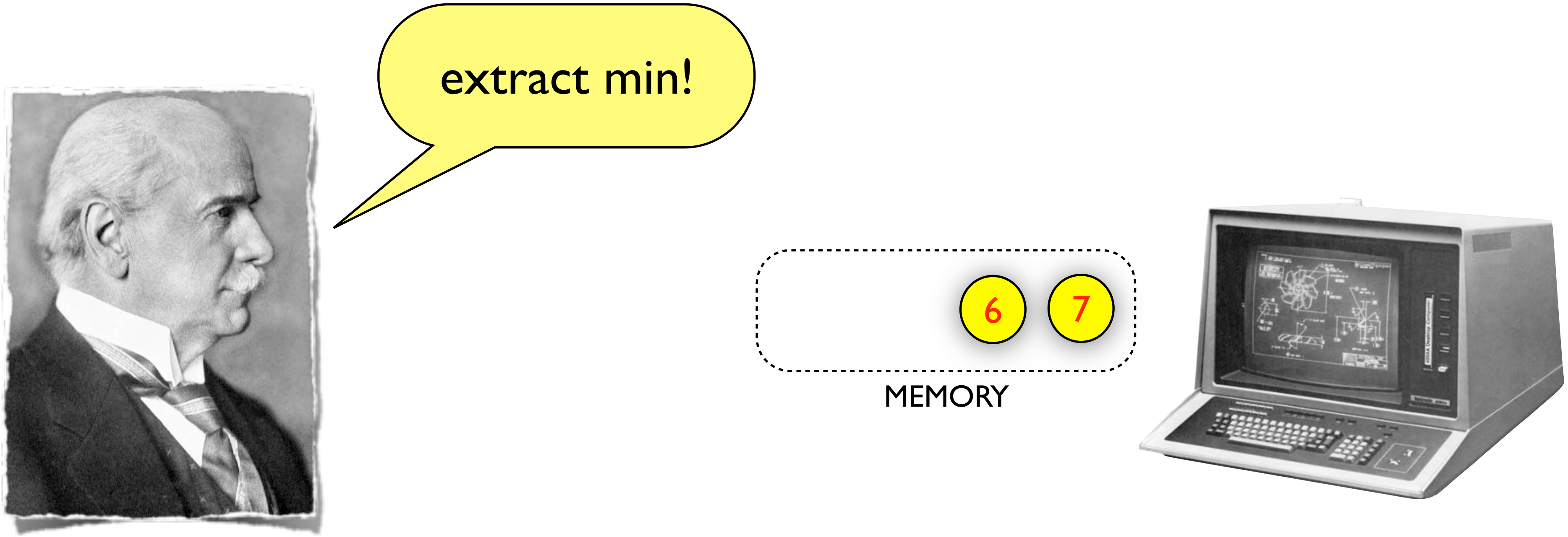

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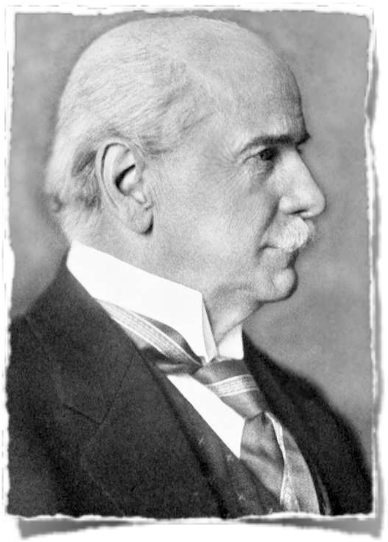

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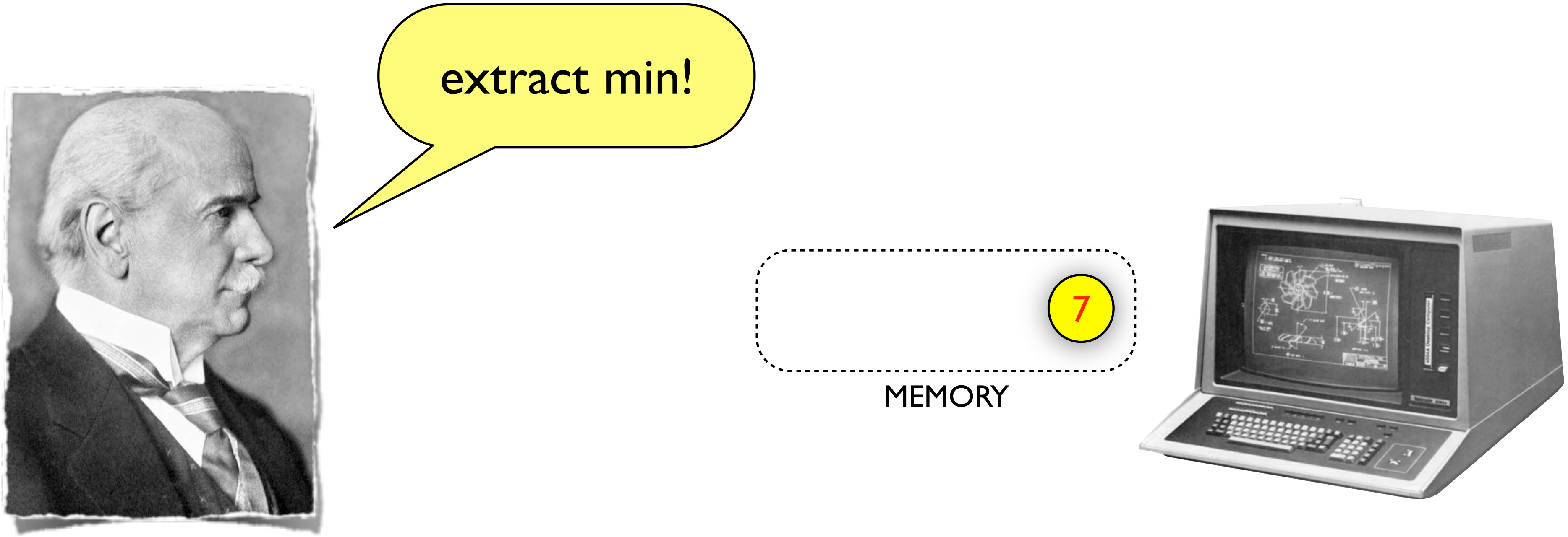

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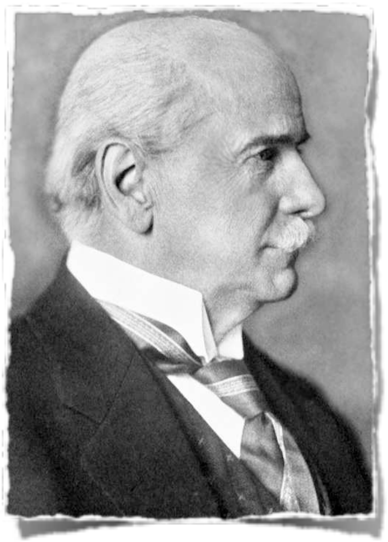

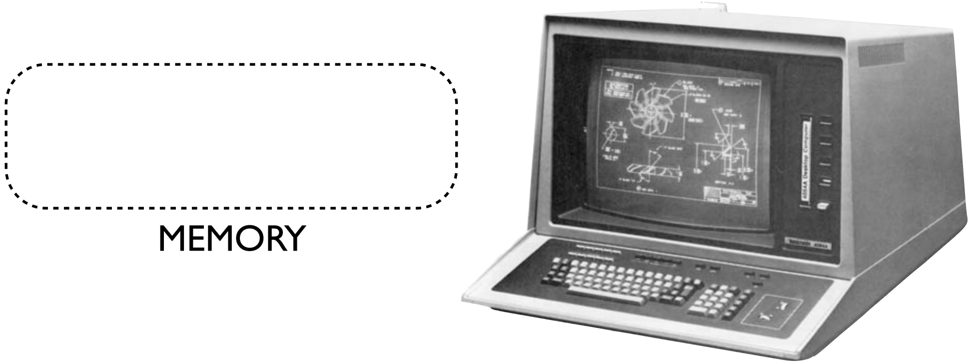

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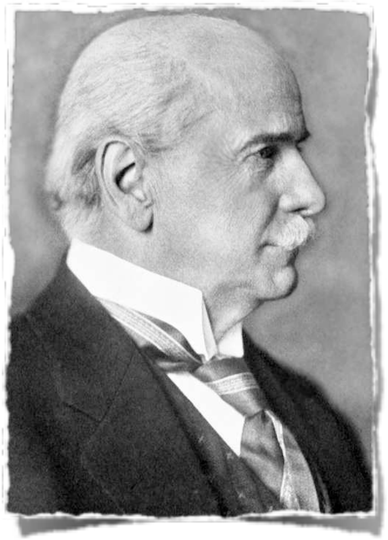

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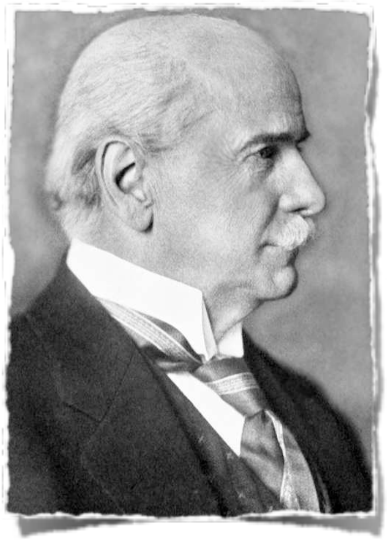

MEMORY

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- *Motivation:* Memory checking useful when using cheap commodity hardware. [Blum, Evans, Gemmell, Kannan, Naor '94]

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? *Big Question: Is annotation necessary for sub-linear space?*

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[Magniez, Mathieu, Nayak '10]

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? Big Question: *Does* $\tilde{O}(\log N)$ *space suffice if we're only allowed multiple forward passes?*

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III. *Information Complexity Trade-offs for Augmented Index*

- Even multi-round protocol leak information.

I. Memory Checking

II. Lower Bounds

III. Augmented Indexing

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! *For this talk:* Assume inserted elements are distinct and that inserts come before their corresponding extract. I.e., we're trying to identify the following *bad pattern:*

ins(u) .... ext(v) ... ext(u) for some u < v

Epochs and Local Bad Patterns…

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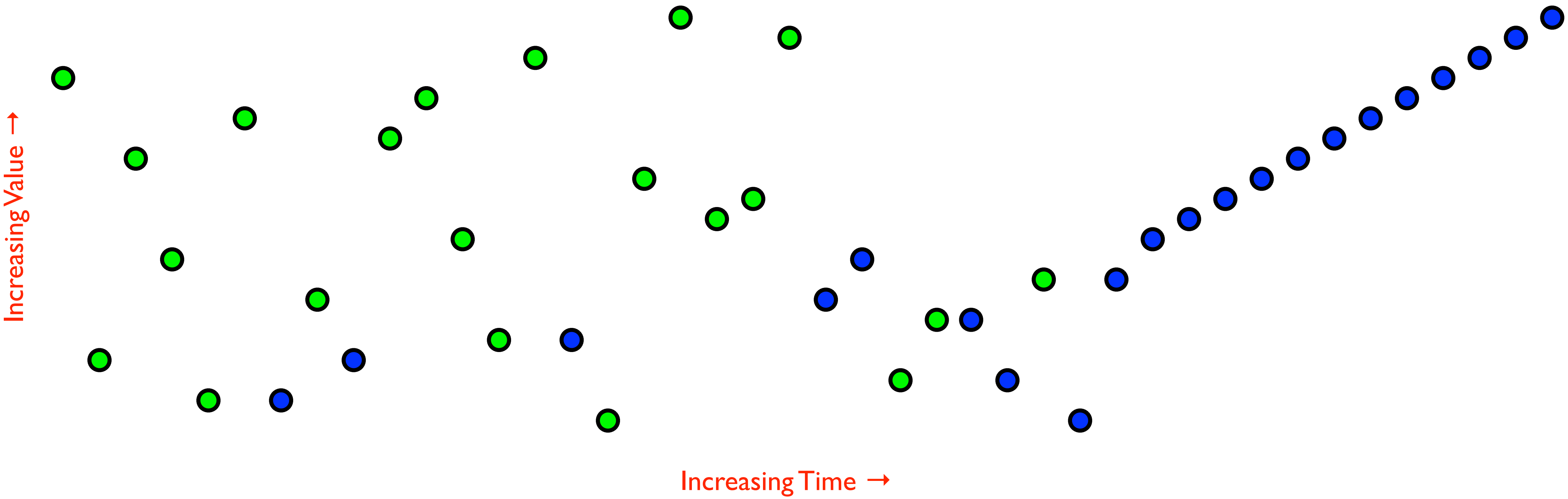

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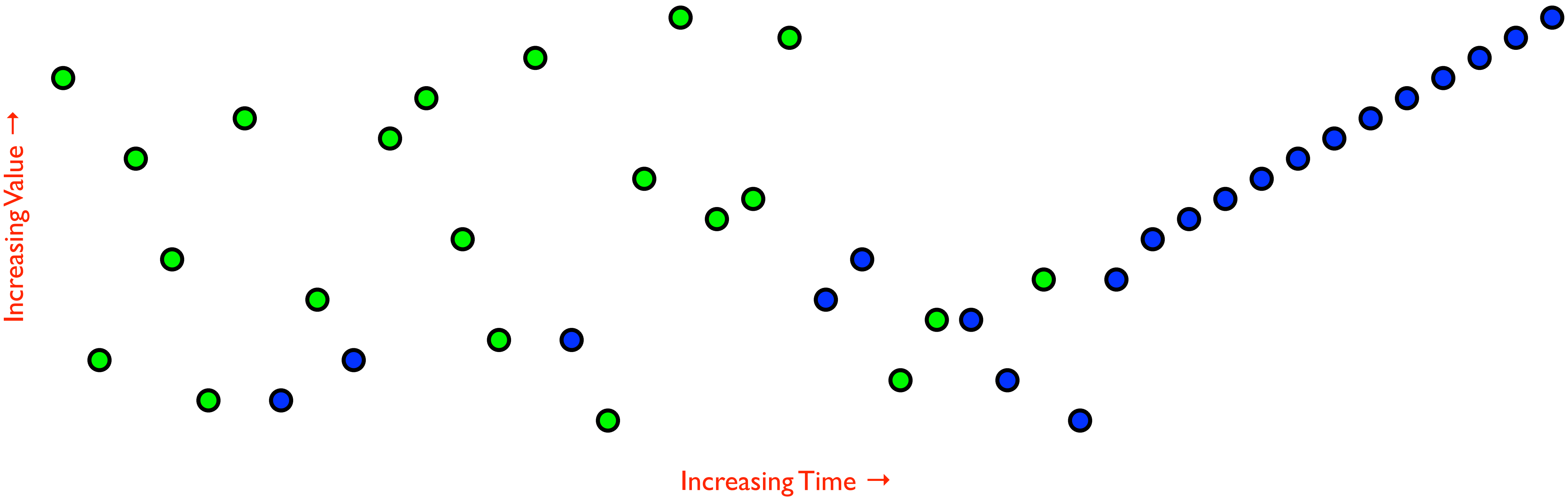

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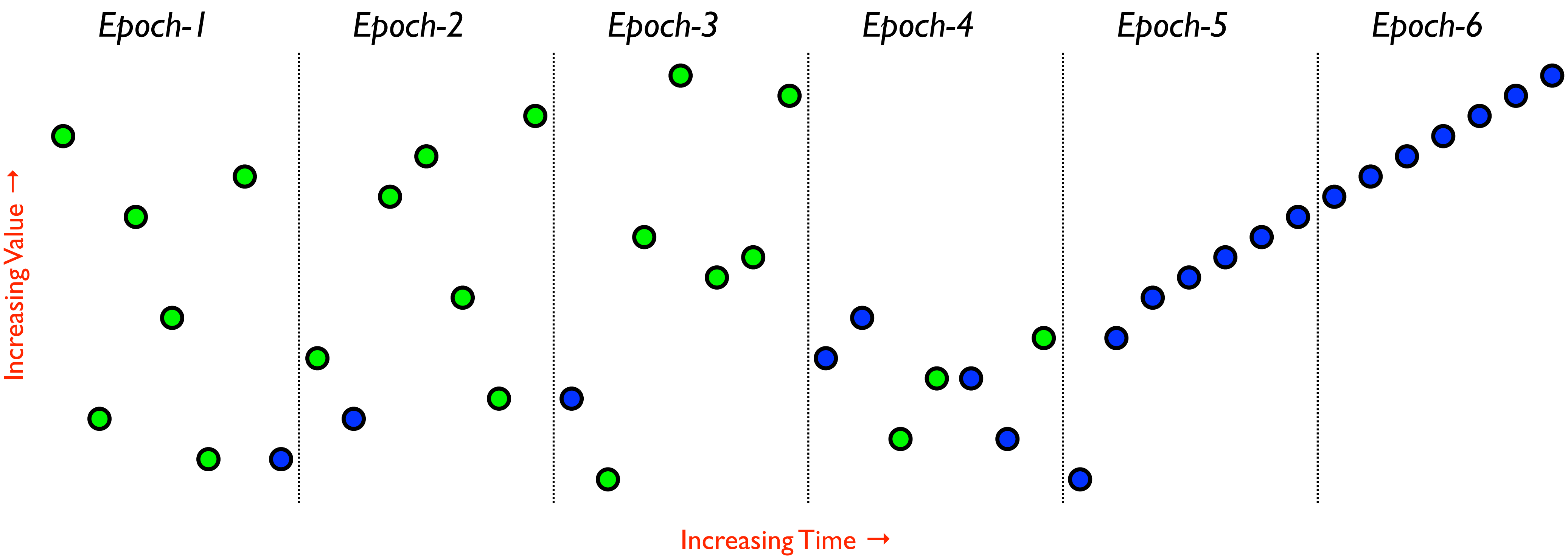

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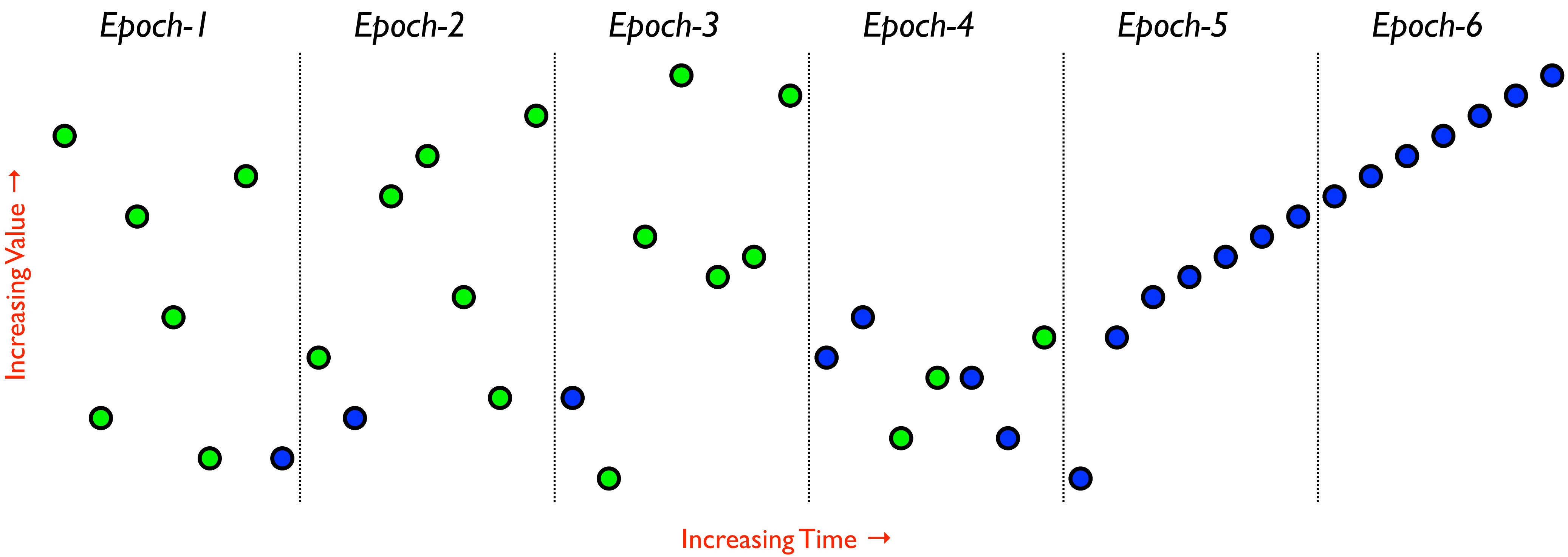

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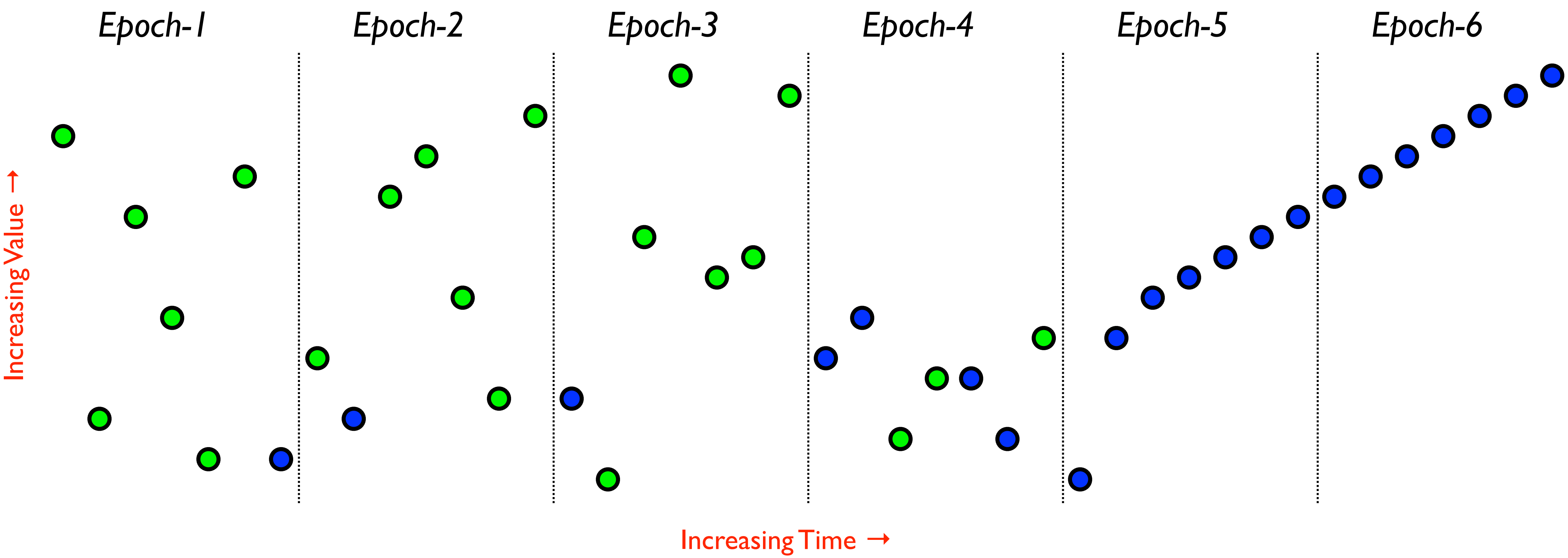

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- Using $O(\sqrt{N})$ space, we can buffer each epoch and check for local bad patterns.

Catching Long-Range Bad Patterns… 1/2

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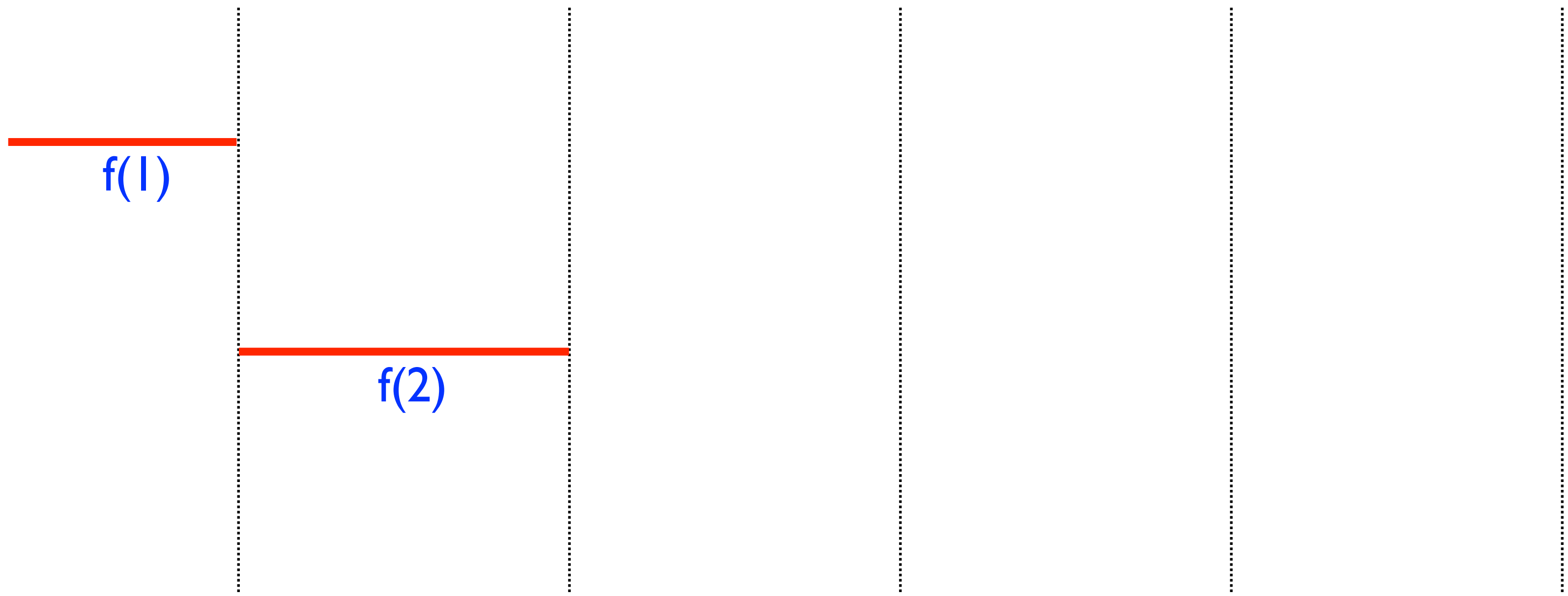

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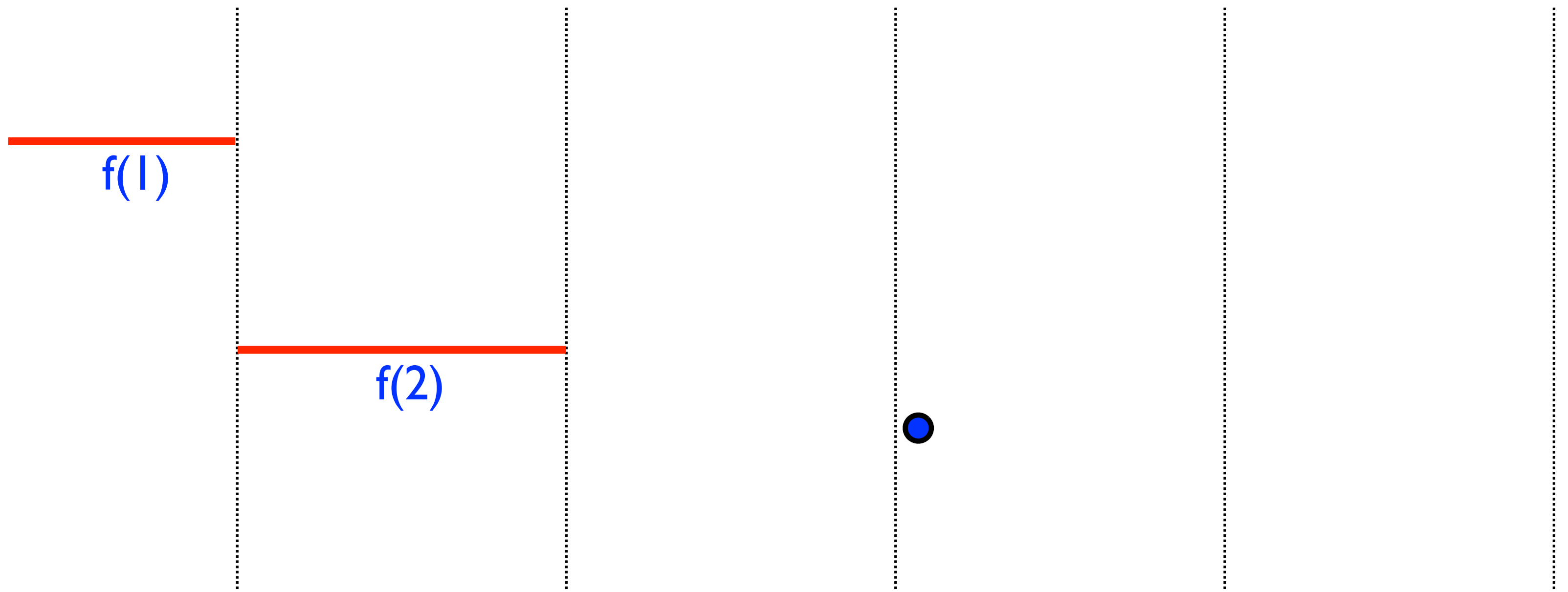

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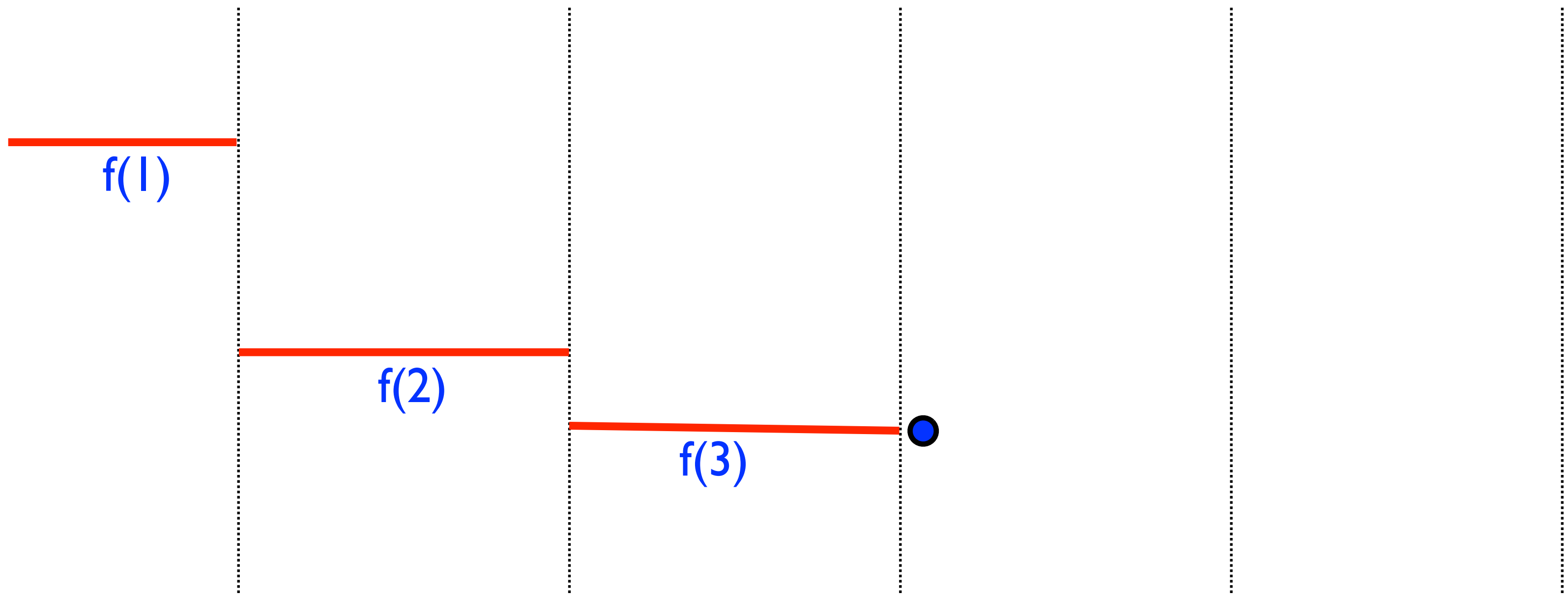

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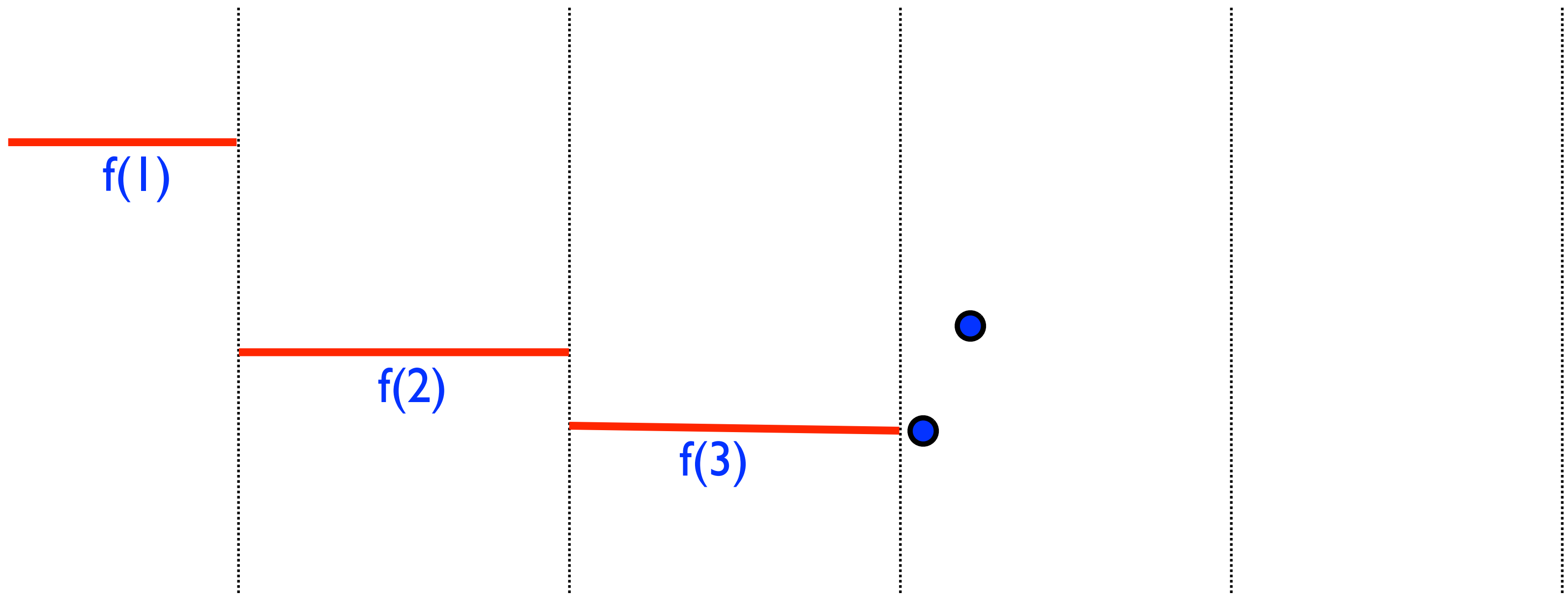

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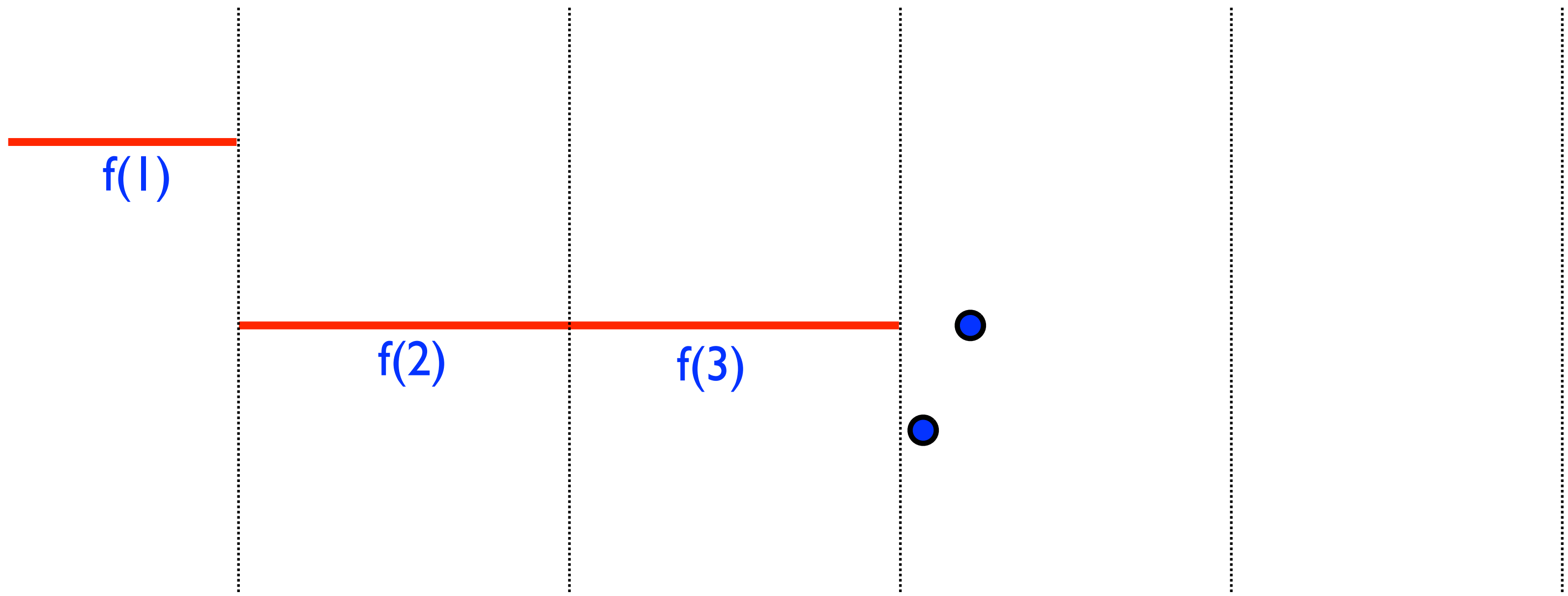

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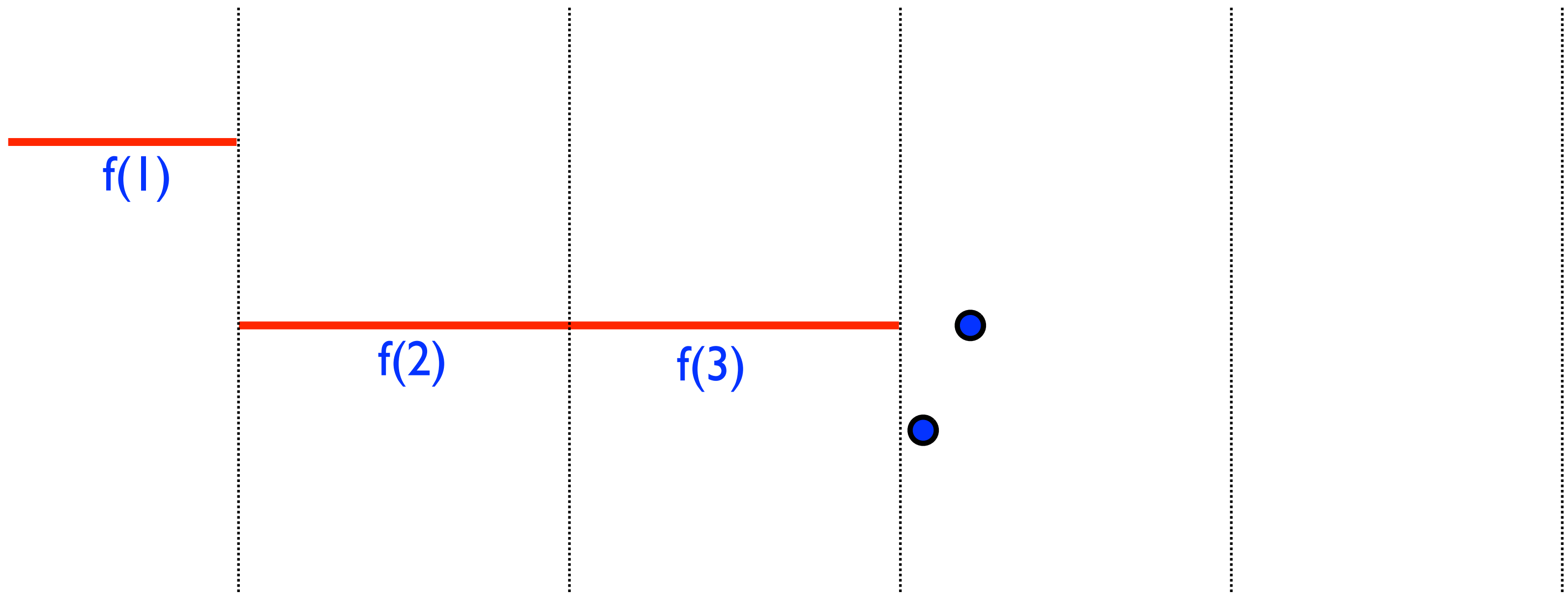

- ***Defn:*** Let $f_t(i)$ be maximum value extracted between end of i-th epoch and *present time* t.
- ***Defn:*** Each insert or extract is *adopted* by k-th epoch where $k = \min\{j : f(j) \leq u\}$ where we assume f(current epoch)=0.

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- *Lemma:* If there are no bad patterns, every ins(u) and ext(u) pair get adopted by the same epoch.
- *Algorithm:* Using fingerprinting to check:

$$\{(u,k) : \text{ins}(u) \text{ adopted by } k\} = \{(u,k) : \text{ext}(u) \text{ adopted by } k\}.$$

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- *Extensions:* Sub-linear space streaming recognition of other data structures like stacks, double-ended queues...

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$y \in \{0,1\}^{k-1}$, $k \in [n]$, $c \in \{0,1\}$

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- *Augmented Index:* Alice has $x \in \{0,1\}^n$ and Bob has a prefix $y \in \{0,1\}^{k-1}$ of x and $c \in \{0,1\}$. Bob wants to check if $c = x_k$.

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- *Augmented Index:* Alice has $x \in \{0,1\}^n$ and Bob has a prefix $y \in \{0,1\}^{k-1}$ of x and $c \in \{0,1\}$. Bob wants to check if $c = x_k$.

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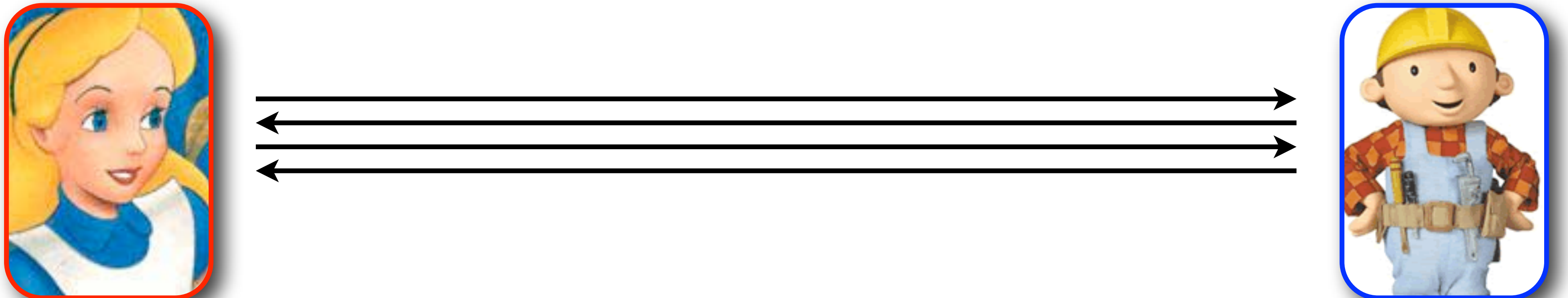
Augmented Index



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- Our main result concerns multi-way protocols but we'll cover the relevance to $DYCK_2$ and PQ first...

Multi-player Augmented Index

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x^1


y^1, k^1, c^1


x^2


y^2, k^2, c^2


x^3


y^3, k^3, c^3

- We now have 2m players $A_1, \ldots, A_m, B_1, \ldots, B_m$ where each A_i and B_i have an instance (x^i, k^i, c^i) of AI_n

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- Want to determine if any of the AI instances are false using private messages communicated in the order

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Multi-player Augmented Index

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- *Thm:* Any 1/3-error, p-round protocol for $\text{MULTI-AI}_{m,n}$ needs $ps = \Omega(\min m, n)$ where s is max message length.

Reduction to Dyck

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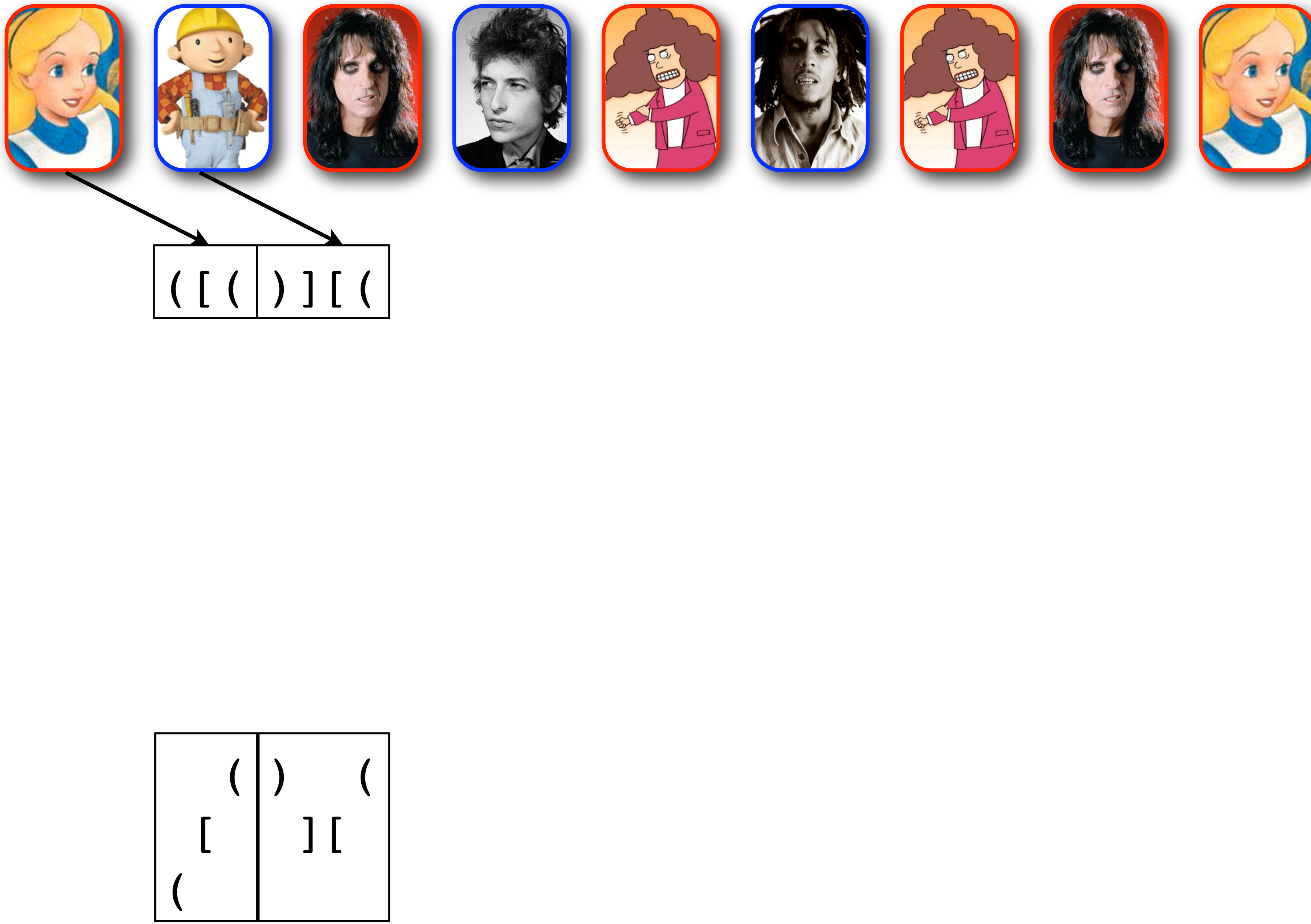

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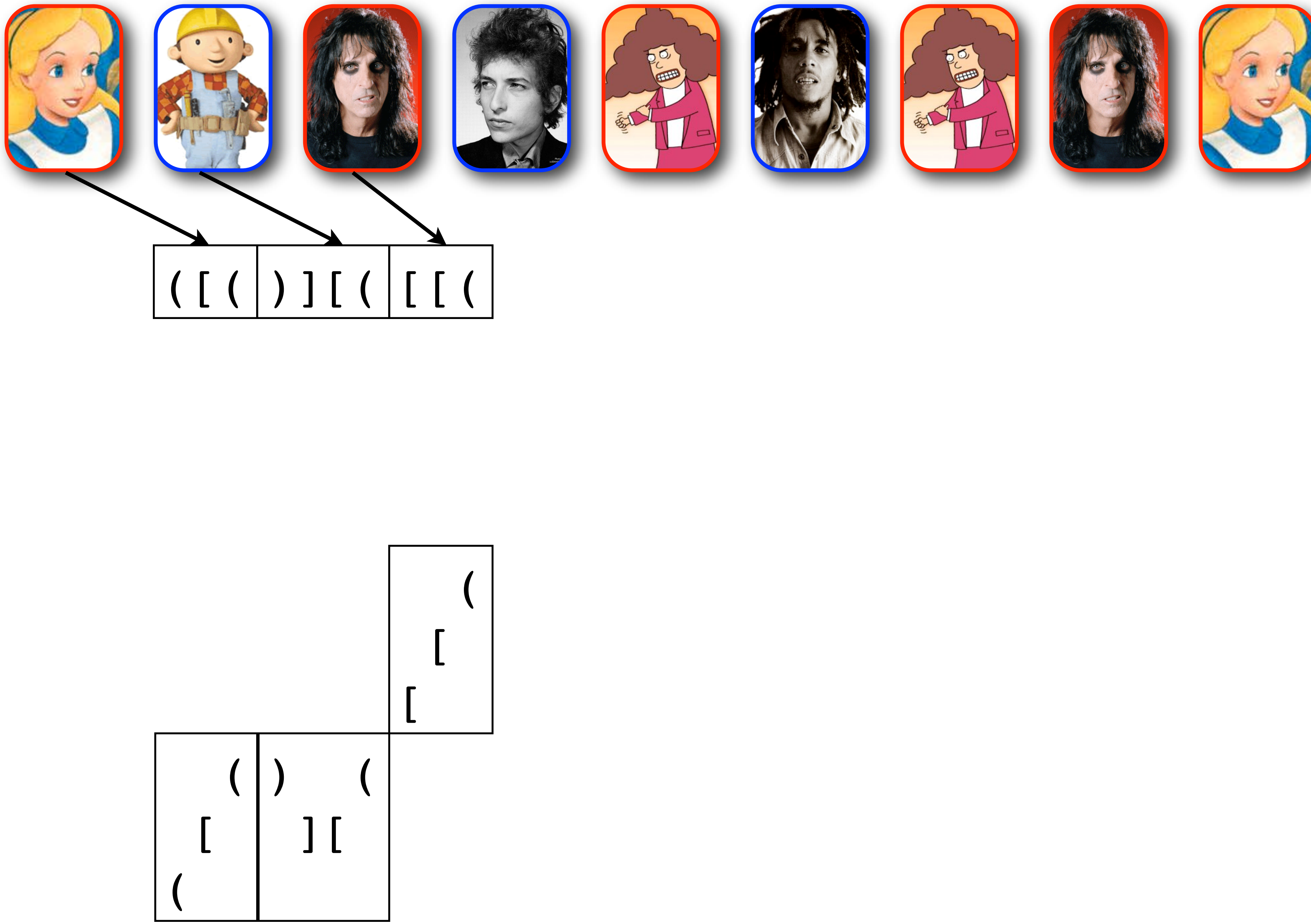

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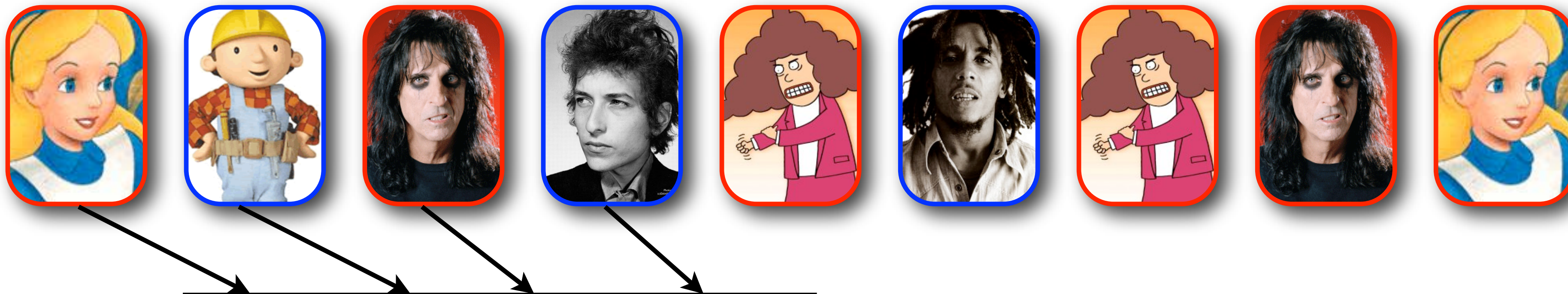

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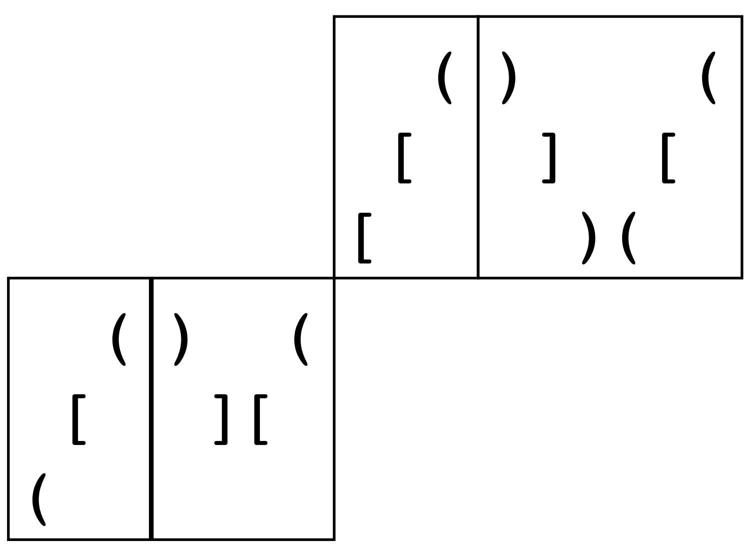

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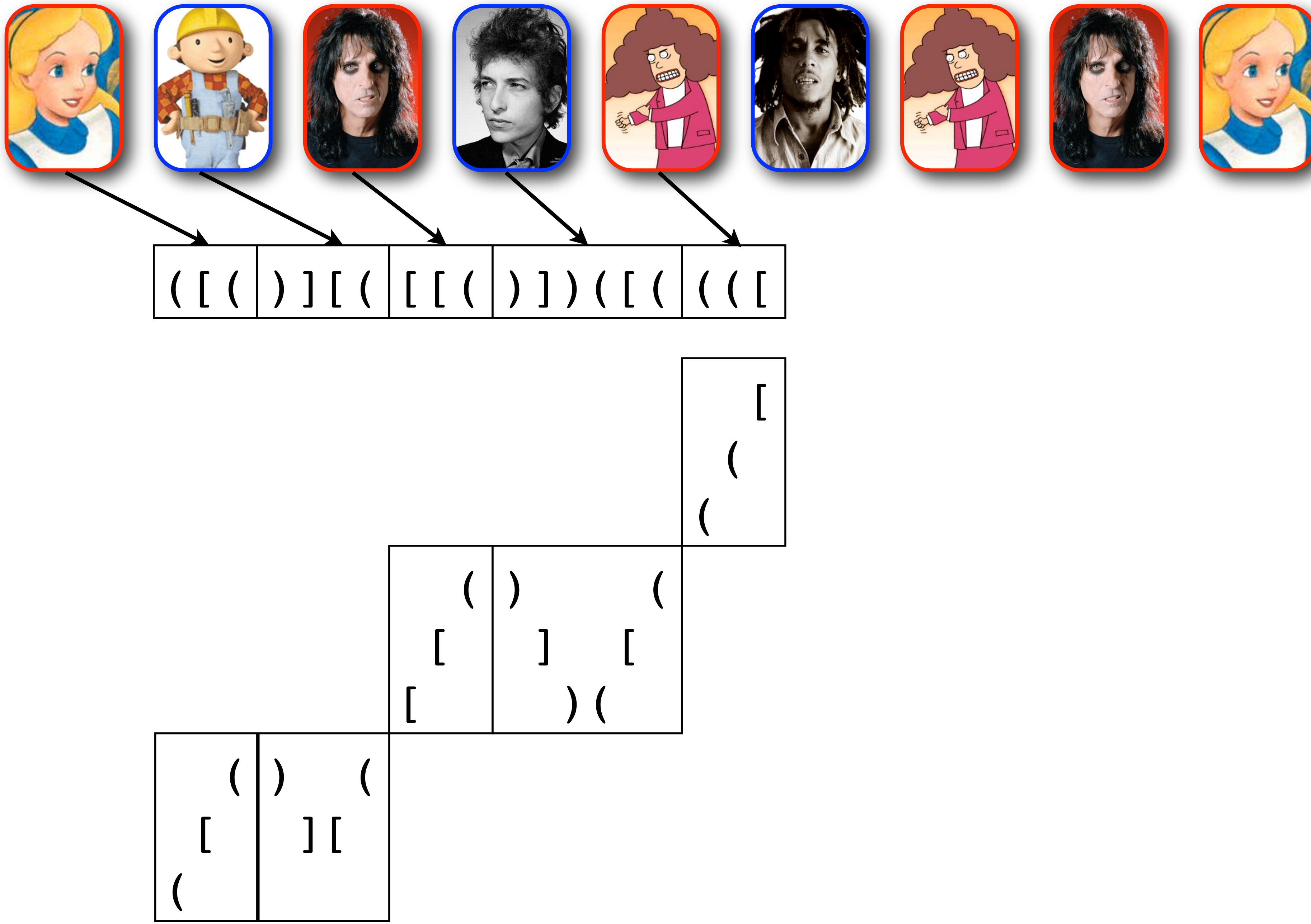

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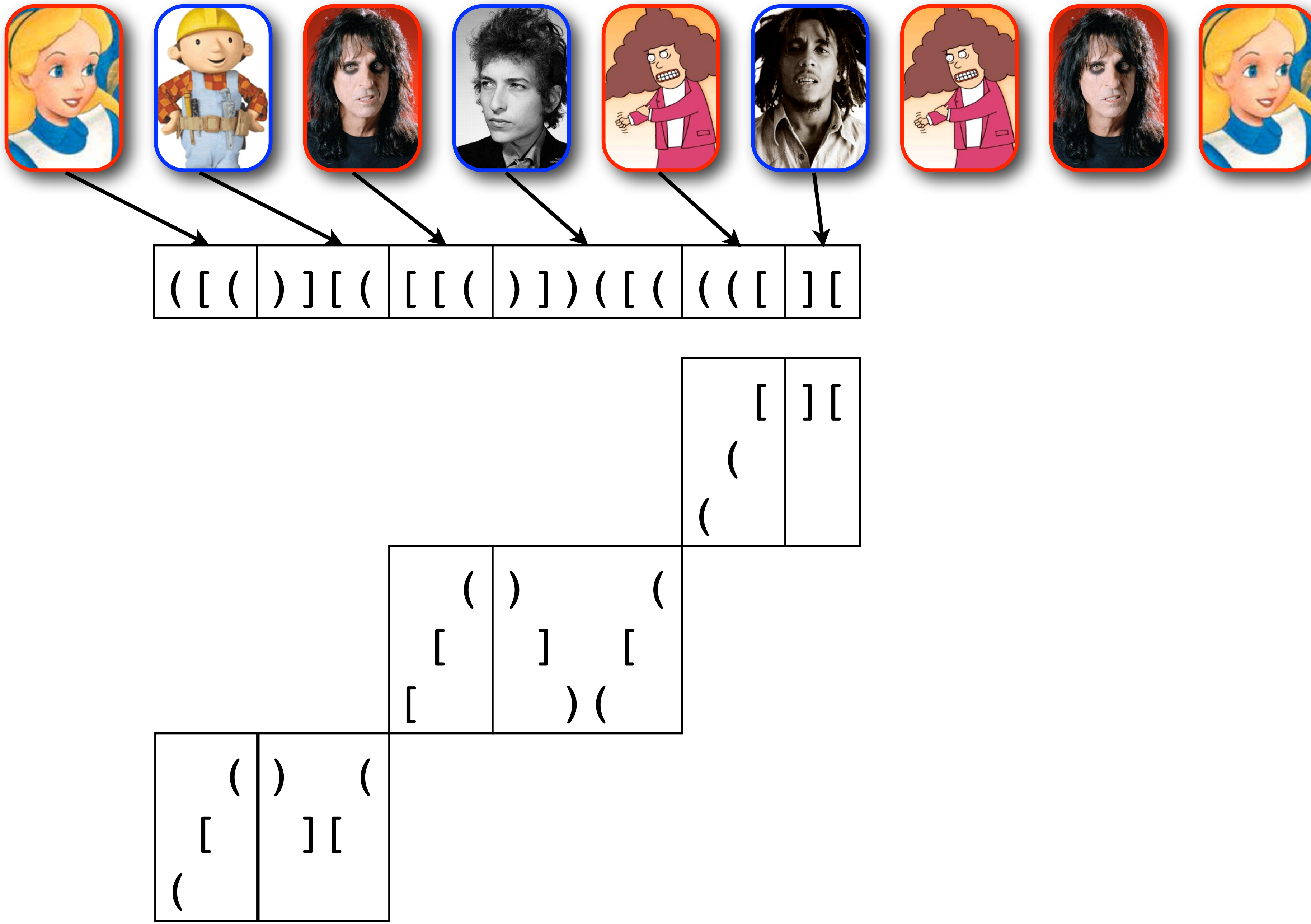

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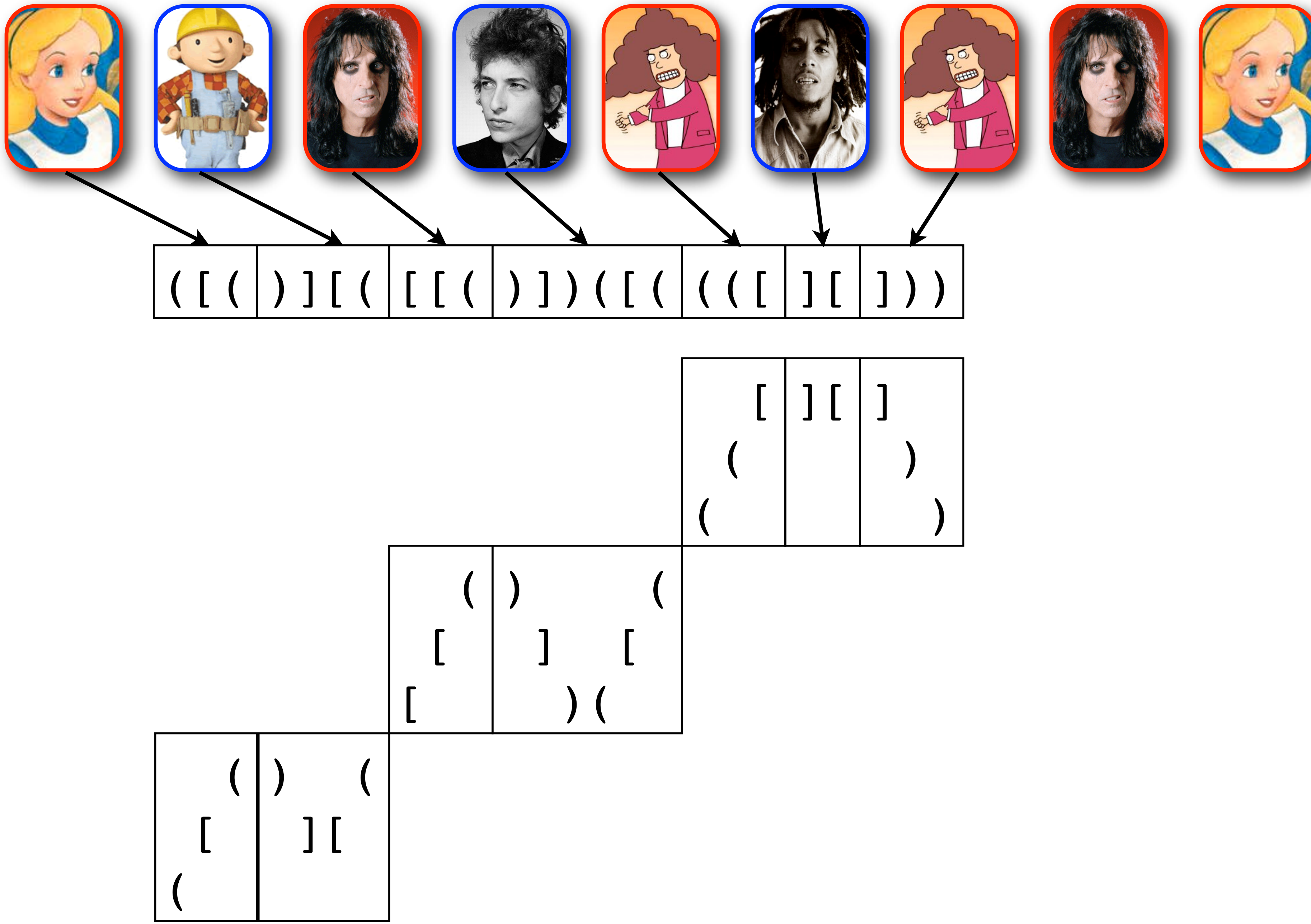

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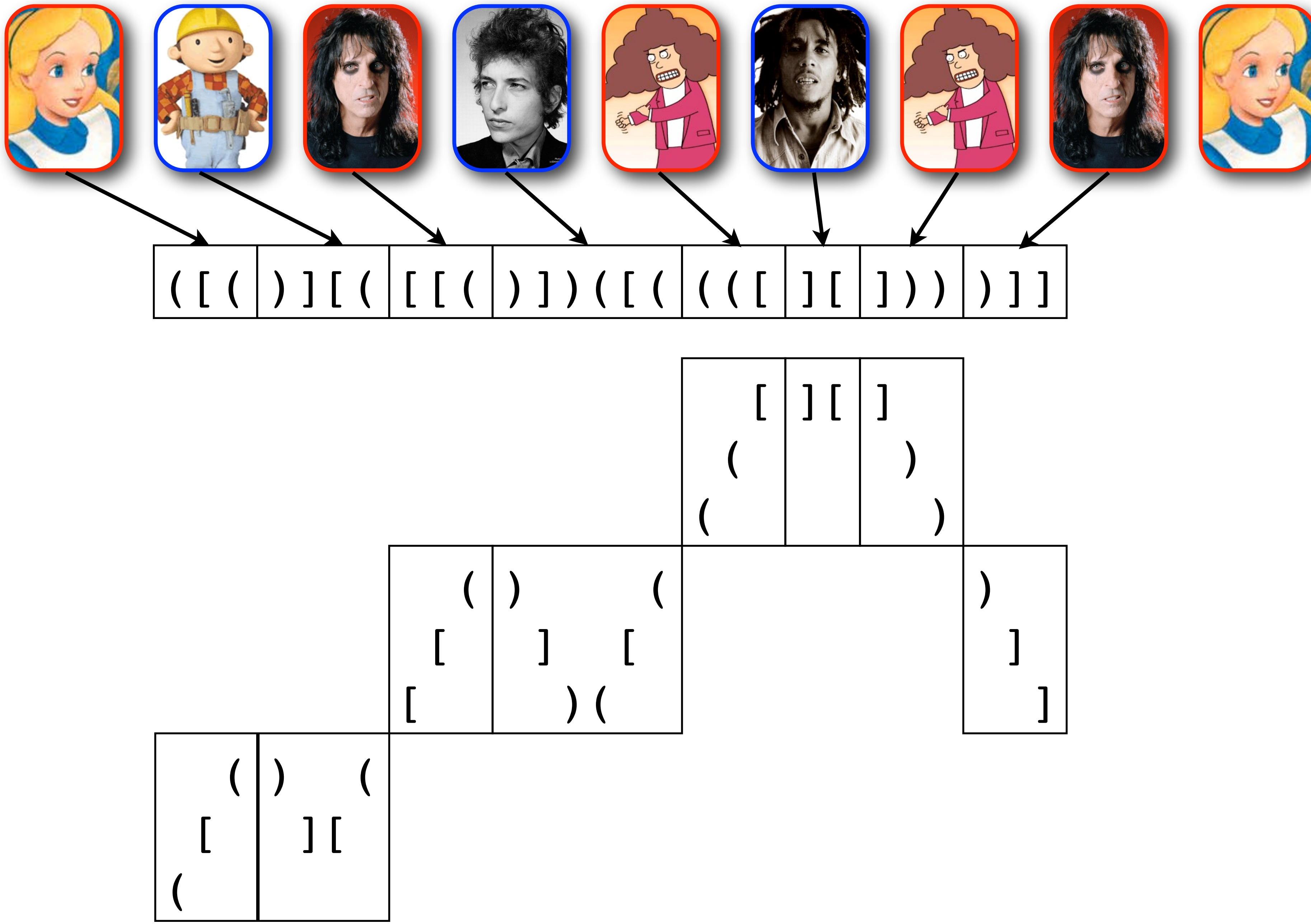

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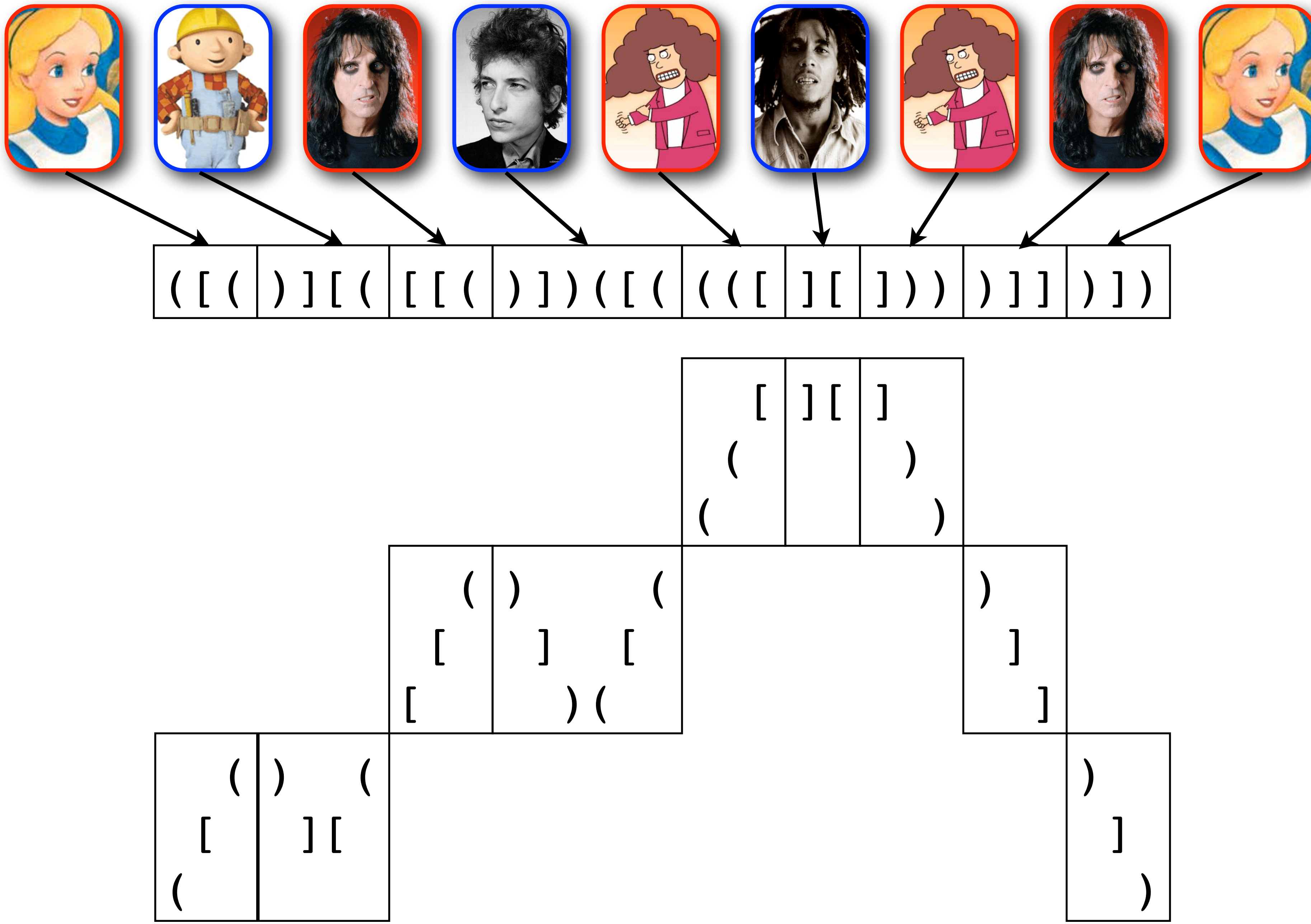

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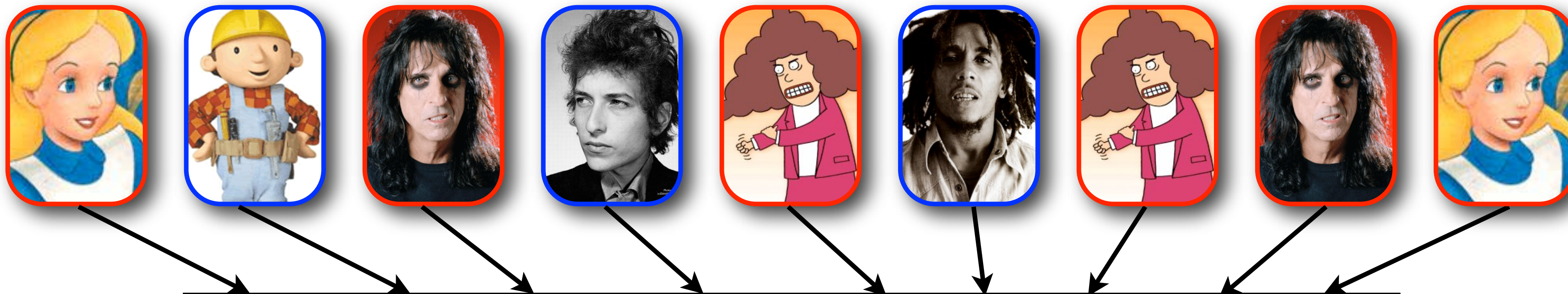

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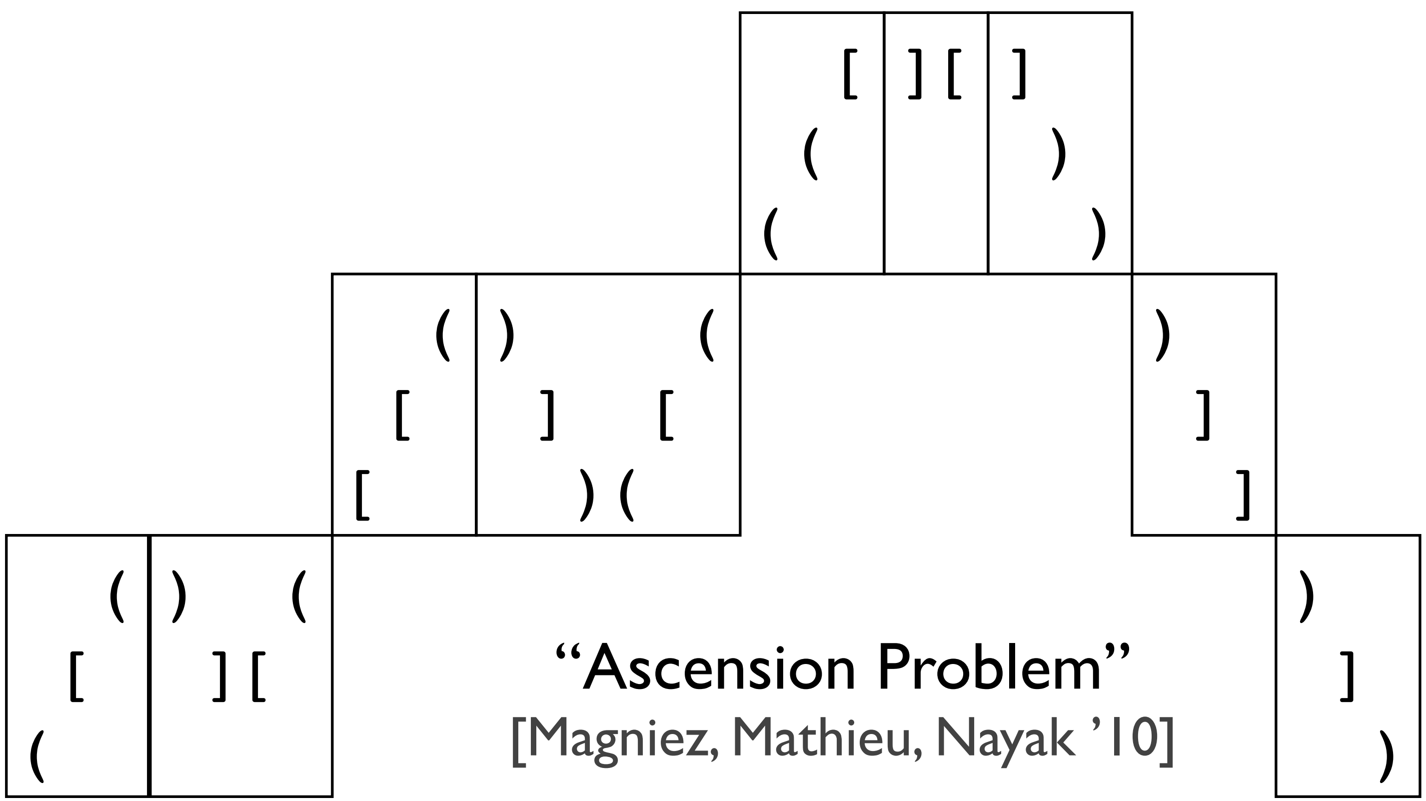

"Ascension Problem"

[Magniez, Mathieu, Nayak '10]

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  - *iii.* Therefore s is $\Omega(\sqrt{N})$ as required.

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- ***Consequences:***

    i. Multiple forward passes have no significant advantage for recognizing the languages considered.

    ii. One forward pass + one reverse pass is exponentially more powerful than two forward passes.

I. Memory Checking

II. Lower Bounds

III. Augmented Indexing

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Information Complexity

[Chakrabarti, Shi, Wirth, Yao '01]

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- ***Entropy and Mutual Information:***

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H(X) &= -\Sigma \Pr[X = x] \lg \Pr[X = x] \\
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- *Information cost method:* Consider mutual information between random input for a communication problem and the communication transcript:

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- Can restrict to partial transcript and subsets of input: useful for proving direct-sum arguments.

Information Complexity of AI_n

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- ***Defn:*** Let P be a protocol for AI_n using public random string R. Let T be the transcript and (X, K, C)~ξ. Define

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\mathrm{icost}^{A}_{\xi}(\mathrm{P}) &= I(T : X \mid K, C, R) \\
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- *Thm:* Let P be a randomized protocol for AI_n with error 1/3 under the uniform distribution μ. Then,

$$\mathrm{icost}^{A}_{\mu_0}(P) = \Omega(n) \quad \text{or} \quad \mathrm{icost}^{B}_{\mu_0}(P) = \Omega(1)$$

where μ_0 is μ conditioned on $X_K=C$.

MULTI-$AI_{m,n}$ versus AI_n

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- ***Defn:*** Let Q be a protocol for MULTI-$AI_{m,n}$ using public random string R. Let T be transcript and $(X^i,K^i,C^i)_{i\in[m]}\sim\xi$.

$$\mathrm{icost}_\xi(Q) = I(T_m : K^1, C^1, \ldots, K^m, C^m \mid X^1, \ldots, X^m, R)$$

where T_m is the set of messages sent by B_m.

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- ***Thm (Direct Sum):*** If there exists a p-round, s-bit, ε-error protocol Q for MULTI-AI$_{m,n}$ then there exists a p-round, ε-error randomized protocol P for AI$_n$ where

i. Alice sends at most ps bits

ii. $m \cdot \text{icost}^B_{\mu_0}(P) \leq \text{icost}_{\mu_0^{\otimes m}}(Q)$

Putting it all together...

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- *Thm:* Any p-round, s-bit, 1/3-error protocol Q for MULTI-$AI_{m,n}$ requires $ps=\Omega(\min m,n)$.

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   ii. By information complexity of AI_n

$$\max(m \cdot \mathrm{icost}^B_{\mu_0}(P), \mathrm{icost}^A_{\mu_0}(P)) = \Omega(\min(m, n))$$

Summary

Memory Checking: Sub-linear space recognition of various data-structure transcript languages is possible without annotation!

Theory of Stream Computation: Forward + reverse pass can be much more useful than many forward passes!

Further Work: Annotations, stream language recognition, ...

Thanks!