Annotation in Data Streams
“with a little help from your friends”

Amit Chakrabarti
Dartmouth College

Graham Cormode
AT&T Research Labs

Andrew McGregor
University of Massachusetts, Amherst
Data Stream Model

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[Alon, Matias, Szegedy '96] [Henzinger, Raghavan, Rajagopalan '98]
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e.g., \([x_1, x_2, \ldots, x_m] = 3, 5, 3, 7, 5, 4, 8, 7, 5, 4, 8, 6, 3, 2, 6, 4, 7, \ldots\)
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- **Goal**: Compute a function of stream, e.g., median, number of distinct elements, longest increasing sequence.

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  i) Limited working memory, i.e., sublinear(n,m)
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• Origins in ’70s but has become popular in last ten years because of growing theory and very applicable.
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• **Today’s talk:** What should model be and how powerful is it?
Model Version 1: “Just trust the helper”

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- Don’t want to have to trust the third party.

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- Helper first announces the answer: verification is easy.
- Can’t expect the helper to know the future.
Model Version 3: “Helper is loquacious”  

cf. “Best-Order Streaming” [Das Sarma, Lipton, Nanongkai 09]

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• Fingerprint to check annotation B is rearranged stream A:

  For prime $q \geq 3m$ and $r \in \mathbb{R}$ [q]: check $\text{FP}_A(r) = \text{FP}_B(r)$ where

  $$\text{FP}_{S}(x) = \prod_{i \in S} (x - i) \mod q$$
Final(ish) Model
Problem: Given stream $S$, want to compute $f(S)$:

$$S = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, \ldots, x_m]$$
Final(ish) Model

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- **Helper**: augments stream with “good” \( h \)-bit annotation:
  \[ (S, a) = [x_1, x_2, x_3, a_3, x_4, x_5, x_6, x_7, a_7, \ldots, x_m, a_m] \]
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- **Verifier:** using $v$ bits of space and random string $r$, run verification algorithm to compute $g(S, a, r)$ such that:
  
  a) $\Pr_r[g(S, a, r) = f(S)] \geq 1 - \delta$
  
  b) $\Pr_r[g(S, a', r) = \perp] \geq 1 - \delta$ for $a' \neq a$
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  b) $\Pr_r[g(S, a', r) = \bot] \geq 1 - \delta$ for $a' \neq a$

- **Goal:** Minimize $h$ and $v$ for given error $\delta$. 
Better Median Protocol

- **Problem:** Find median of \( m \) numbers from \([m]\).
- **Thm:** \( O(\sqrt{m \log m}) \) annotation bits and \( O(\sqrt{m \log m}) \) verification memory is sufficient to find the median.
Upper Bound...
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- Define “cumulative frequency” vector: \( g_i = | \{ j : x_j \leq i \} | \)
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\begin{array}{cccccccccccccccccccc}
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- Define “cumulative frequency” vector: \( g_i = |\{j : x_j \leq i\}| \)

- Easy to see \( i \) is median iff \( g_{i-1} < m/2 \) and \( g_i \geq m/2 \)
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Easy to see \( i \) is median iff \( g_{i-1} < \frac{m}{2} \) and \( g_i \geq \frac{m}{2} \)

Partition \( g \) into \( v = \sqrt{m} \) segments of length \( h = \sqrt{m} \)
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```
1 1 3 5 6 7 8 8 8 10 10 11 12 12 15 18 18 19 20 22 22 23 25 25
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- Easy to see \( i \) is median iff \( g_{i-1} < m/2 \) and \( g_i \geq m/2 \)
- Partition \( g \) into \( v = m^{1/2} \) segments of length \( h = m^{1/2} \)
- **Verifier:** a) Construct fingerprint of each segment
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- Partition \( g \) into \( v = m^{1/2} \) segments of length \( h = m^{1/2} \)
- **Verifier:**
  a) Construct fingerprint of each segment
  b) Compute last entry in each segment
Define “cumulative frequency” vector: 
\[ g_i = |\{ j : x_j \leq i \}| \]

Easy to see \( i \) is median iff 
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Partition \( g \) into \( v = \frac{m}{\sqrt{2}} \) segments of length \( h = \frac{m}{\sqrt{2}} \)

**Verifier:**

a) Construct fingerprint of each segment
b) Compute last entry in each segment
c) Identify “interesting” segment
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- Partition \( g \) into \( v = \frac{m}{2} \) segments of length \( h = \frac{m}{2} \)

- **Verifier:**
  a) Construct fingerprint of each segment
  b) Compute last entry in each segment
  c) Identify “interesting” segment

- **Helper:** Presents entirety of interesting segment
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- **Thm:** Any protocol for median requires 
  \[(\text{annotation “} h \text{”}) \times (\text{verification memory “} v \text{”}) = \Omega(m).\]
Lower Bound ...
Lower Bound ...

- Suppose algorithm “A” has parameters \((h,v)\), error \(\delta = 1/3\)
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Define “B”:

\[\text{ALGORITHM “B”}\]
1. Run \(t = \Theta(h)\) copies of “A” in parallel
2. Use annotation \(a\) for all copies
3. Output majority answer if it exists, else \(\perp\)

Algorithm “B” has parameters \((h,hv)\), error \(\delta = (1/3)2^{-h}\)

If \(a\) valid, then expect \((2/3)t\) runs to return median

If \(a\) not valid, then expect \((2/3)t\) runs give \(\perp\)
**Lower Bound ...**

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- **Define “B”:**
  
  **ALGORITHM “B”**
  
  1. Run \(t = \Theta(h)\) copies of “A” in parallel
  2. Use annotation \(a\) for all copies
  3. Output majority answer if it exists, else \(\perp\)

- Algorithm “B” has parameters \((h,hv)\), error \(\delta = (1/3)2^{-h}\)
  
  If \(a\) valid, then expect \((2/3)t\) runs to return median
  
  If \(a\) not valid, then expect \((2/3)t\) runs give \(\perp\)

- **Define “C”:** Verifier ignores annotation, tries all \(2^h\) possible annotations and ensures error at most 1/3 (by union bound).
Suppose algorithm “A” has parameters \((h,v)\), error \(\delta = 1/3\)

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Algorithm “C” solves median of stream using \(O(hv)\) space
Our Results

- **Median**: Optimal trade-off of annotation & verification.
- **Frequency Moments**: Optimal trade-off for exact version and lower bounds for approximate version.
- **Heavy Hitters**: Optimal trade-off for exact and protocol for approximate version via CM-sketch verification.
- **Graph Problems**: Trade-offs for counting triangles, matchings, and connectivity. Optimal in some regimes.
1. Frequency Moments
2. Counting Triangles
3. Beyond the Moraines...
1. Frequency Moments
2. Counting Triangles
3. Beyond the Moraines...
Frequency Moments

- **Problem:** Given $m$ numbers from $[n]$, compute

$$F_k = \sum_{i \in [n]} f_i^k$$

where $f_i = \text{freq. of } i$
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  *using “algebrization” ideas from* [Aaronson, Widgerson 08]*]
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• **Thm:** Any protocol for \( F_k \) requires

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(\text{annotation}) \times (\text{verification memory}) = \Omega(n)
\]

and any constant factor approx. requires

\[
(\text{annotation}) \times (\text{verification memory}) = \Omega(n^{1-5/k}).
\]

  *using ideas from* [Klauck 03], [Alon, Mattias, Szegedy 99]*
Upper Bound (1/2)...


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- Transform universe $[n]$ into $[\sqrt{n}] \times [\sqrt{n}]$: $\sum_{i,j \in [\sqrt{n}]} f_{i,j}^k$
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\[
\begin{bmatrix}
1 & 0 & 3 \\
2 & 1 & 10 \\
9 & 8 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 3 \\
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9 & 8 & 3
\end{bmatrix}
\]

"frequency vector"  
"frequency square"
Upper Bound (1/2)...

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  \]

- Define \(f(x, y) \in \mathbb{F}_q[x, y]\) where \(q\) large prime:
  \[
  f(x, y) = \sum_{(i,j) \in S} p_{i,j}(x, y) \quad \text{where} \quad p_{i,j}(x, y) = \prod_{\ell=1: \ell \neq i}^{\sqrt{n}} \frac{x - \ell}{j - \ell} \cdot \prod_{\ell=1: \ell \neq j}^{\sqrt{n}} \frac{y - \ell}{j - \ell}
  \]

<table>
<thead>
<tr>
<th>1</th>
<th>0</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
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</tr>
<tr>
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f(i, j) = \text{frequency of } (i, j) \text{ in stream}
\]

- **Observe:** \(f\) defined incrementally and:

\[
\deg_x(f) = \deg_y(f) = \sqrt{n} - 1
\]
Upper Bound (2/2)...
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- **Verifier:** pick $r \in \mathbb{R}[q]$; compute $f(r,y)$; and determine

$$C = \sum_{j \in [\sqrt{n}]} f(r, j)^{k}$$
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Lower Bound...
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- **Idea:** Use MA lower bound for 2-party set-disjointness.
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  \]

- **Idea:** New MA lower bound for \( t \)-party set-disjointness.
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2. Counting Triangles
3. Beyond the Moraines...
Counting Triangles
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• **Problem:** Given \( m \) edges on \( n \) nodes, find \# triangles.
Counting Triangles

- **Problem:** Given $m$ edges on $n$ nodes, find the number of triangles.

- **Thm:** $O(n^{3\alpha} \log n)$ annotation bits and $O(n^{3-3\alpha} \log n)$ verification memory suffices for $0 < \alpha < 1$.

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  Construct induced stream $S$ by replacing each $(i,j)$ by $(i,j,k)$ for each $k \neq i,j$. Compute $(F_3(S) - 2F_2(S) + F_1(S)) / 12$. 
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- **Thm:** Triangles requires \((\text{annotation}) \times (\text{verification}) = \Omega(n^2)\).
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Interactive Proofs & Streams

Forthcoming work from [Cormode, Yi '09]...
Interactive Proofs & Streams
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- *Motivated by applications clouding computing:*

  1. Consider messages back and forth between prover and verifier after the stream has been observed.

  2. Measure in terms of memory used by verifier and subsequent communication.
Interactive Proofs & Streams
Forthcoming work from [Cormode, Yi '09]...

• Motivated by applications clouding computing:
  1. Consider messages back and forth between prover and verifier after the stream has been observed.
  2. Measure in terms of memory used by verifier and subsequent communication.

• Results:
  1. Using log memory and log communication, many database problems can be solved exactly: self join size, frequency moments, range queries, index etc.
Sketch Verification
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- **Sketch:** Let $A$ be $k$ by $n$ measurement matrix and compute $Af$: for stream $[1, 2, 3, 2, 4,...]$,

  $$Af = a_1 + a_2 + a_3 + a_2 + a_4 + ...$$ where $a_i$ is $i^{\text{th}}$ column of $A$
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• **Annotation Protocol:** Helper gives $Af$ in $O(k \log n)$ bits and verifier checks in $O(\log k)$ bits by composing fingerprint with $A$. Let $B$ be $k'$ by $k$ and compute
  
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- **Tricky Part:** Having helper guide verifier through the steps of extracting information from sketch.
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- **Tricky Part**: Having helper guide verifier through the steps of extracting information from sketch.

- **Mysterious Issue**: Most useful sketches are random but can’t trust helper to pick random bits.
**Summary**

**Model:** Merlin-Arthur communication meets data streams and intractable stream problems become solvable!

**Results:** Annotation/verification trade-offs for classical stream problems. Some optimal and some open questions.

Thanks!