CMPSCI 711: More Advanced Algorithms

Vectors 2: Sketching F_0 and F_2

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Hash Functions

Definition

A family $\mathcal{H}$ of functions from $A \to B$ is k-wise independent if for any distinct $x_1, \ldots, x_k \in A$ and $i_1, i_2, \ldots, i_k \in B$,

$$\mathbb{P}_{h \in_R \mathcal{H}}[h(x_1) = i_1, h(x_2) = i_2, \ldots, h(x_k) = i_k] = \frac{1}{|B|^k}$$

Example

Suppose $A \subset \{0, 1, 2, \ldots, p-1\}$ and $B = \{0, 1, 2, \ldots, p-1\}$. Then,

$$\mathcal{H} = \{h(x) = \sum_{i=0}^{k-1} a_i x^i \bmod p : 0 \leq a_0, a_1, \ldots, a_{k-1} \leq p-1\}$$

is a k-wise independent family of hash functions.

Note. If $|B|$ is not prime or $|A| > |B|$ more ideas are required.

Linear Sketches

- A sketch algorithm stores a random matrix $Z \in \mathbb{R}^{k \times n}$ where $k \ll n$ and computes projection Zf of the frequency vector.
- *Can be computed incrementally:*
  - Suppose we have sketch Zf of current frequency vector f.
  - If we see an occurrence of i, the new frequency vector is $f' = f + e_i$
  - Can update sketch be just adding i column of Z to Zf:

$$Zf' = Z(f + e_i) = Zf + Ze_i = Zf + (i\text{-th column of } Z)$$

- *Useful?* Need to choose random matrices such that relevant properties of f can be estimated with high probability from Zf.

Outline

F_2 Estimation

Distinct Elements

F_2

- *Problem:* Construct an (ϵ, δ) approximation for $F_2 = \sum_i f_i^2$
- *Algorithm:*
  - Let $Z \in \{-1, 1\}^{k \times n}$ where entries of each row are 4-wise independent and rows are independent.
  - Compute Zf and average squared entries appropriately.
- *Analysis:*
  - Let $s = z.f$ be an entry of Zf where z is a row of Z.
  - *Lemma:* $\mathbb{E}\left[s^2\right] = F_2$
  - *Lemma:* $\mathbb{V}\left[s^2\right] \leq 4F_2^2$

Expectation Lemma

- $s = z.f$ where $z_i \in_R \{-1, 1\}$ are 4-wise independent.
- Then

$$\mathbb{E}\left[s^2\right] = \mathbb{E}\left[\sum_{i,j \in [n]} z_i z_j f_i f_j\right] = \sum_{i,j \in [n]} f_i f_j \mathbb{E}\left[z_i z_j\right] = \sum_{i \in [n]} f_i^2$$

since $\mathbb{E}\left[z_i z_j\right] = 0$ unless $i = j$,

Variance Lemma

- $\mathbb{E}\left[z_i z_j z_k z_l\right] = 0$ unless $(i,k) = (j,l)$, $(i,j) = (k,l)$ or $(i,j) = (l,k)$
- Then

$$\begin{aligned}
\mathbb{V}\left[s^2\right] = \mathbb{E}\left[s^4\right] - \mathbb{E}\left[s^2\right]^2 &= \sum_i f_i^4 + 6\sum_{i<j} f_i^2 f_j^2 - (\sum_{i\in[n]} f_i^2)^2 \\
&= 4\sum_{i<j} f_i^2 f_j^2 \\
&\leq 4F_2^2
\end{aligned}$$

Averaging "Appropriately"

- Group entries of the sketch into $a = O(\log \delta^{-1})$ groups of $b = 12\epsilon^{-2}$
- Let $Y_1, Y_2, \ldots, Y_a$ be the average of squared entries in each group.

$$\mathbb{E}\left[Y_i\right] = F_2$$

$$\mathbb{V}\left[Y_i\right] \leq 4F_2^2/b$$

- By Chebychev, $\mathbb{P}\left[|Y_i - F_2| \geq \epsilon F_2\right] \leq \frac{4F_2^2}{b(\epsilon F_2)^2} = 1/3$
- By Chernoff, median$(Y_1, \ldots, Y_a)$ is a (ϵ, δ) approximation of F_2.

Extension to Estimating ℓ_p

- The ℓ_p norm is defined as $\ell_p(f) = (\sum_i |f_i|^p)^{1/p}$
- A distribution D is p-stable if given $X, Y \sim D$ and $a, b \in \mathbb{R}$ then

$$aX + bY \sim (a^p + b^p)^{1/p} D$$

- E.g., Cauchy and Gaussian distributions are 1 and 2-stable:

$$\text{Cauchy}(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} \quad \text{Gaussian}(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

- If entries of matrix $z_{i.j} \sim D$ are p stable, then projection entries:

$$s \sim \ell_p(f) D$$

- For $p \in (0, 2]$, can (ϵ, δ) estimate ℓ_p in $O(\epsilon^{-2} \operatorname{polylog}(n, m))$ space.

Outline

F_2 Estimation

Distinct Elements

Distinct Elements

- *Problem:* Construct an (ϵ, δ) approximation for $F_0 = \sum_i f_i^0$
- *Simpler problem:* For given $T > 0$, with probability $1 - \delta$ distinguish between $F_0 > (1 + \epsilon)T$ and $F_0 < (1 - \epsilon)T$
- If we can solve simpler problem, can solve original problem by trying $O(\epsilon^{-1} \log n)$ values of T

$$T = 1, (1 + \epsilon), (1 + \epsilon)^2, \ldots, n$$

- *Algorithm:*
  - Choose random sets $S_1, S_2, \ldots, S_k \subset [n]$ where $\mathbb{P}[i \in S_j] = 1/T$
  - Compute $s_j = \sum_{i \in S_j} f_i$
  - If at least k/e of the s_j are zero, output $F_0 < (1 - \epsilon)T$
- *Analysis:*
  - If $F_0 > (1 + \epsilon)T$, $\mathbb{P}[s_j = 0] < 1/e - \epsilon/3$
  - If $F_0 < (1 - \epsilon)T$, $\mathbb{P}[s_j = 0] > 1/e + \epsilon/3$
  - Chernoff: $k = O(\epsilon^{-2} \log \delta^{-1})$ ensures correctness with prob. $1 - \delta$.

Analysis

- Suppose T is large and ϵ is small:

$$\mathbb{P}\left[s_j = 0\right] = (1 - 1/T)^{F_0} \approx e^{-F_0/T}$$

- If $F_0 > (1+\epsilon)T$,

$$e^{-F_0/T} \leq e^{-(1+\epsilon)} \leq e^{-1} - \epsilon/3$$

- If $F_0 < (1-\epsilon)T$,

$$e^{-F_0/T} \geq e^{-(1-\epsilon)} \geq e^{-1} + \epsilon/3$$