CMPSCI 711: More Advanced Algorithms

Graphs 7: Multi-Pass Matchings via Multiplicative Weights

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- A $(1+\epsilon)$ approx for max cardinality matching in bipartite graphs.

[Ahn, Guha ICALP 11]

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Today's Result

Theorem

There's a $O(\epsilon^{-4} n \,\text{polylog}\, n)$-space algorithm, using $O(\epsilon^{-3} \log n)$ passes that $1+\epsilon$ approximates max cardinality matching in bipartite graphs.

The results follows from the following lemma by trying values $\alpha = 1, (1+\epsilon), (1+\epsilon)^2 \ldots n$ in parallel.

Lemma

There's a $O(\epsilon^{-3} \log n)$-pass, $O(\epsilon^{-4} n \,\text{polylog}\, n)$-space algorithm that given value α either:

1. *Finds a matching of size $\frac{\alpha}{1+4\epsilon}$*
2. *Or determines the maximum matching has size at most $\alpha(1+2\epsilon)$.*

Ideas behind algorithm: Fractional Matchings

Fact

For any graph, the size of max fractional matching equals for the size of the min fractional vertex cover, i.e.,

$$\max\left\{\sum_{(i,j)\in E} y_{ij} : \sum_{j\in\Gamma(i)} y_{ij} \leq 1 \ \forall i \in V\right\} = \min\left\{\sum_{i\in V} x_i : x_i + x_j \geq 1 \ \forall (i,j) \in E\right\}$$

For a bipartite graph, this also equals the size of max (integral) matching.

Ideas behind algorithm: Multiplicative Weights

- y is a fractional matching if for all $i \in V$, $M_i(y) \leq 0$ where

$$M_i(y) = \left(\sum_{j \in \Gamma(i)} y_{ij} \right) - 1 \ .$$

- Rather than trying to satisfy all n constraints, it's easier to satisfy them "on average", i.e., given a distribution μ over $[n]$ find y with

$$\sum_i \mu_i M_i(y) \leq 0 \ .$$

- The algorithm defines sequence of distributions $\mu^1, \mu^2, \mu^3, \ldots, \mu^T$ and finds a set of weights $y^1, y^2, \ldots, y^T$ for each.
- If $M_i(y^t) > 0$ then μ^{t+1} increases weight on that constraint such that average of $y^1, \ldots, y^T$ nearly satisfies all constraints.

Algorithm and Analysis

1. Define $\mu_i^1 = 1/n$ and $u_i^1 = 1$ for all $i \in V$
2. For $t = 1$ to T:
   2.1 **Construct y^t:** Let $x_i = \alpha\mu_i^t$ and find maximal matching S in
   $$\{(i,j) \in E : x_i + x_j < 1\}$$
   If $|S| < \delta\alpha$ output "Fail" and otherwise define
   $$y_{ij}^t = \alpha/|S| \text{ for each } (i,j) \in S \text{ and } 0 \text{ otherwise.}$$
   2.2 **Define μ^{t+1}**: $u_i^{t+1} = u_i^t(1 + \epsilon\delta M_i(y^t))$ and $\mu_i^{t+1} = \frac{u_i^{t+1}}{\sum_i u_i^{t+1}}$ $\forall i \in V$.
3. Output $\frac{1}{1+4\epsilon}\sum_t y^t/T$.

Lemma

If fail, max fractional matching is $< (1+2\delta)\alpha$. Otherwise, y^t satisfies:

$$\sum_{i \in V} \mu_i M_i(y^t) \leq 0 \qquad \sum_{e \in E} y_e = \alpha \qquad -1 \leq M_i(y^t) \leq 1/\delta \ \ \forall i \in V$$

Theorem

Output is fractional matching size $\alpha/(1+4\epsilon)$ if $T = O(\epsilon^{-3}\log n)$, $\delta = \epsilon$.

Proof of Lemma

- If we fail, for each i that is an endpoint of an edge in S, consider adding 1 to each x_i. Then gives a fractional vertex cover of size

$$\sum_i \alpha\mu_i^t + 2|S| < \alpha + 2\delta\alpha$$

and hence that max fractional matching has size $< \alpha(1+2\delta)$.

- To prove y^t satisfies constraints on average

$$\sum_{i\in V} x_i \left(\sum_{j\in\Gamma(i)} y_{ij} - 1 \right) = \sum_{i\in V} x_i \sum_{j\in\Gamma(i)} y_{ij} - \sum_i x_i = \sum_{ij\in E} y_{ij}(x_i + x_j) - \alpha < \alpha - \alpha$$

where last inequality follows since, $x_i + x_j < 1$ for $(i,j) \in S$.

- To prove size of y^t is α,

$$\sum_{e\in E} y_e = \sum_{e\in S} \alpha/|S| = \alpha$$

- To prove upper and lower bounds on $M_i(y^t)$,

$$-1 \le \sum_{j\in\Gamma(i)} y_{ij}^t - 1 < \alpha/|S| \le 1/\delta\ .$$

Proof of Theorem

- $\sum_i u_i^t$ decreases as t increases:

$$\sum_i u_i^{t+1} = \sum_i u_i^t(1+\delta\epsilon M_i(y^t)) = \sum_i u_i^t + \delta\epsilon \sum_i u_i M_i(y^t) \leq \sum_i u_i^t$$

- Use this to bound u_i^{T+1}:

$$n \geq \sum_i u_i^{T+1} \geq u_i^{T+1} = \prod_{t\in[T]} (1+\delta\epsilon M_i(y^t))$$

- After some algebra and using fact $-1 \leq M_i(y^t) \leq 1/\delta$:

$$\ln n \geq (\epsilon - \epsilon^2)\delta \sum_{t\in[T]} M_i(y^t) - 2\epsilon^2 T\delta$$

- Finally, setting $y = \sum_t y^t/T$ and $T = \frac{\ln n}{\delta\epsilon^2}$ gives,

$$\sum_{j\in\Gamma(i)} y_{ij} - 1 = M_i(y) = \sum_{t\in[T]} \frac{M_i(y^t)}{T} \leq \frac{\ln n + 2\epsilon^2 T\delta}{T\delta(\epsilon-\epsilon^2)} \leq 4\epsilon$$

and therefore $y' = y/(1+4\epsilon)$ satisfies $\sum_{j\in\Gamma(i)} y'_{ij} \leq 1$ for all $i \in [n]$.

Missing Algebra for Proof of Theorem

- Using $1+\epsilon x \geq \begin{cases} (1+\epsilon)^x & \text{if } 0 \leq x \leq 1 \\ (1-\epsilon)^{-x} & \text{if } -1 \leq x \leq 0 \end{cases}$ we get:

$$n \geq \prod_{t\in[T]} (1+\delta\epsilon M_i(y^t)) \geq (1+\epsilon)^{\delta \sum_{t:M_i(y^t)>0} M_i(y^t)}(1-\epsilon)^{-\delta \sum_{t:M_i(y^t)<0} M_i(y^t)}$$

- Taking logs of both sides gives:

$$\begin{aligned}
\ln n &\geq \ln(1+\epsilon) \sum_{t:M_i(y^t)>0} \delta M_i(y^t) - \ln(1-\epsilon) \sum_{t:M_i(y^t)<0} \delta M_i(y^t) \\
&\geq (\epsilon-\epsilon^2) \sum_{t:M_i(y^t)>0} \delta M_i(y^t) + (\epsilon+\epsilon^2) \sum_{t:M_i(y^t)<0} \delta M_i(y^t) \\
&= (\epsilon-\epsilon^2) \sum_{t\in[T]} \delta M_i(y^t) + 2\epsilon^2 \sum_{t:M_i(y^t)<0} \delta M_i(y^t) \\
&\geq (\epsilon-\epsilon^2) \sum_{t\in[T]} \delta M_i(y^t) - 2\epsilon^2 T\delta
\end{aligned}$$