CMPSCI 711: More Advanced Algorithms

Graphs 6: Small Matchings

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Overview:

- An exact algorithm using $O(k^2 \log k)$ space for finding the largest cardinality matching in the insert-delete model where k is an upper bound on the largest matching size.

[Chitnis et al. SODA 16]

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Small Matching

Theorem

Suppose match$(G) \leq k$. *There exists a* $O(k^2 \log k)$ *space algorithm in the insert-delete model that finds the size of the largest matching.*

Algorithm:

- Let $c : [n] \to [b]$ be a 2-wise hash function where $b = 1000k$.
- For each $i, j \in [b]$, recover a single edge $\{x, y\}$ (if one exists) with

$$\{c(x), c(y)\} = \{i, j\}$$

- Repeat $O(\log k)$ times in parallel and return the largest matching amongst the recovered edges.

Subgraphs with same size max matching

Lemma

Let $match(G) \leq k$ *and* G' *be a subgraph of* G*. Let*

$$U = \{u : \deg_G(u) \geq 10k\} \text{ and } F = \{e \in E : e \cap U = \emptyset\}$$

Then $match(G) = match(G')$ *if* $F \subseteq G'$ *and* $\deg_{G'}(u) \geq 5k$ *for all* $u \in U$.

Proof.

- $|U| \leq 2k$ since the min vertex cover has size $\leq 2match(G) \leq 2k$ and every node in U most be in the min vertex cover.
- G' contains a matching of size

  $$match(F) + |U|$$

  since even after we pick the largest matching in F, every node in U has $\geq 5k - 2k - 2k = k$ neighbors in $V \setminus (U \cup match(F))$.
- Max matching in G has size at most $match(F) + |U|$.

□

$\mathbb{P}[F \subseteq G' \text{ and } \deg_{G'}(u) \geq 5k \; \forall u \in U] \geq 1 - \frac{1}{\text{poly}(k)}$.

- Let c be 2-wise independent hash function and H be a graph with one edge $\{x, y\}$ with $c(x) = i, c(y) = j$ for each $i, j \in [b]$.

Claim
If $e \in F$ then $\mathbb{P}[e \in H] \geq 1/2$.

Claim
If $u \in U$ then $\mathbb{P}[\deg_H(u) \geq 5k] \geq 1/2$

- Repeat $r = \Theta(\log k)$ times, to boost probabilities to $1 - \frac{1}{\text{poly}(k)}$.
- Take union bound over $|F| = O(k^2)$ edges and $|U| = O(k)$ nodes.
- The fact $|F| = O(k^2)$ follows since $k \geq match(F) \geq |F|/(10k)$.

Claim 1: $\mathbb{P}[e \in H] \geq 1/2$ for $e \in F$

- Let C be a min vertex cover of G and note $|C| \leq 2k$ because the endpoints of the edges in a maximum matching form a vertex cover.
- Let $e = \{x, y\}$ and consider $A = (C \cup \Gamma(x) \cup \Gamma(y)) \setminus \{x, y\}$
- Then $G[V \setminus A]$ consists of the unique edge e. So if no vertices in A receive hash values equal to $c(x)$ and $c(y)$, then e is unique edge with hash values $c(x)$ and $c(y)$ and hence is in H.
- Since $b = 1000k$ and $|A| \leq 2k + 10k + 10k = 22k$,

$$\begin{aligned}\mathbb{P}[e \in H] &\geq 1 - \mathbb{P}[\exists a \in A : c(a) = c(x)] - \mathbb{P}[\exists a \in A : c(a) = c(y)] \\ &\geq 1 - 2|A|/b > 1/2 \ .\end{aligned}$$

Claim 2: $\mathbb{P}[\deg_H(u) \geq 5k] \geq 1/2$ for $u \in U$

- Let $A = C \setminus \{u\}$. Then $G[V \setminus A]$ is star with center u and $\geq 9k$ leaves. Let $N = \{v_1, \ldots, v_{9k}\}$ be arbitrary set of $9k$ such leaves.
- Let $X_i = 1$ if v_i has the same hash value as some other vertex in N or a vertex in C. Let $X = \sum X_i$.
- If $c(u) \notin c(A)$ and $X \leq 4k$, then H has $\geq 5k$ edges incident to u.
- This happens with probability at least $1/2$ since

$$\mathbb{P}[c(u) \in c(A)] \leq |A|/b < 2k/b = 1/500 \ ,$$

and

$$\mathbb{E}[X_i] \leq \frac{|A| + |N|}{b} \leq \frac{2k + 9k}{b} \leq 1/50 \ ,$$

and so

$$\mathbb{P}[X \geq 4k] \leq \frac{\mathbb{E}[X]}{4k} \leq \frac{9k \times 1/50}{4k} < 1/20 \ .$$