Overview:

▶ An exact algorithm using $O(k^2 \log k)$ space for finding the largest cardinality matching in the insert-delete model where $k$ is an upper bound on the largest matching size.

[Chitnis et al. SODA 16]
Small Matching

Theorem
Suppose $\text{match}(G) \leq k$. There exists a $O(k^2 \log k)$ space algorithm in the insert-delete model that finds the size of the largest matching.

Algorithm:
- Let $c : [n] \rightarrow [b]$ be a 2-wise hash function where $b = 1000k$.
- For each $i, j \in [b]$, recover a single edge $\{x, y\}$ (if one exists) with $\{c(x), c(y)\} = \{i, j\}$
- Repeat $O(\log k)$ times in parallel and return the largest matching amongst the recovered edges.
Subgraphs with same size max matching

Lemma

Let \( \text{match}(G) \leq k \) and \( G' \) be a subgraph of \( G \). Let

\[
U = \{u : \deg_G(u) \geq 10k\} \quad \text{and} \quad F = \{e \in E : e \cap U = \emptyset\}
\]

Then \( \text{match}(G) = \text{match}(G') \) if \( F \subseteq G' \) and \( \deg_{G'}(u) \geq 5k \) for all \( u \in U \).

Proof.

- \( |U| \leq 2k \) since the min vertex cover has size \( \leq 2\text{match}(G) \leq 2k \) and every node in \( U \) must be in the min vertex cover.
- \( G' \) contains a matching of size

\[
\text{match}(F) + |U|
\]

since even after we pick the largest matching in \( F \), every node in \( U \) has \( \geq 5k - 2k - 2k = k \) neighbors in \( V \setminus (U \cup \text{match}(F)) \).
- Max matching in \( G \) has size at most \( \text{match}(F) + |U| \).
\[ P[F \subseteq G' \text{ and } \deg_{G'}(u) \geq 5k \ \forall u \in U] \geq 1 - \frac{1}{\text{poly}(k)}. \]

- Let \( c \) be 2-wise independent hash function and \( H \) be a graph with one edge \( \{x, y\} \) with \( c(x) = i, c(y) = j \) for each \( i, j \in [b] \).

**Claim**

If \( e \in F \) then \( P[e \in H] \geq 1/2 \).

**Claim**

If \( u \in U \) then \( P[\deg_H(u) \geq 5k] \geq 1/2 \)

- Repeat \( r = \Theta(\log k) \) times, to boost probabilities to \( 1 - \frac{1}{\text{poly}(k)} \).
- Take union bound over \( |F| = O(k^2) \) edges and \( |U| = O(k) \) nodes.
- The fact \( |F| = O(k^2) \) follows since \( k \geq \text{match}(F) \geq |F|/(10k) \).
Claim 1: $\mathbb{P} [e \in H] \geq 1/2$ for $e \in F$

- Let $C$ be a min vertex cover of $G$ and note $|C| \leq 2k$ because the endpoints of the edges in a maximum matching form a vertex cover.
- Let $e = \{x, y\}$ and consider $A = (C \cup \Gamma(x) \cup \Gamma(y)) \setminus \{x, y\}$
- Then $G[V \setminus A]$ consists of the unique edge $e$. So if no vertices in $A$ receive hash values equal to $c(x)$ and $c(y)$, then $e$ is unique edge with hash values $c(x)$ and $c(y)$ and hence is in $H$.
- Since $b = 1000k$ and $|A| \leq 2k + 10k + 10k = 22k$, 

$$\mathbb{P} [e \in H] \geq 1 - \mathbb{P} [\exists a \in A : c(a) = c(x)] - \mathbb{P} [\exists a \in A : c(a) = c(y)] \geq 1 - 2|A|/b > 1/2.$$
Claim 2: $\Pr[\deg_H(u) \geq 5k] \geq 1/2$ for $u \in U$

- Let $A = C \setminus \{u\}$. Then $G[V \setminus A]$ is star with center $u$ and $\geq 9k$ leaves. Let $N = \{v_1, \ldots, v_{9k}\}$ be arbitrary set of $9k$ such leaves.
- Let $X_i = 1$ if $v_i$ has the same hash value as some other vertex in $N$ or a vertex in $C$. Let $X = \sum X_i$.
- If $c(u) \not\in c(A)$ and $X \leq 4k$, then $H$ has $\geq 5k$ edges incident to $u$.
- This happens with probability at least $1/2$ since

$$
\Pr[c(u) \in c(A)] \leq \frac{|A|}{b} < \frac{2k}{b} = 1/500,
$$

and

$$
\mathbb{E}[X_i] \leq \frac{|A| + |N|}{b} \leq \frac{2k + 9k}{b} \leq 1/50,
$$

and so

$$
\Pr[X \geq 4k] \leq \frac{\mathbb{E}[X]}{4k} \leq \frac{9k \times 1/50}{4k} < 1/20.
$$