Overview:

- An exact algorithm using $O(k^2 \log k)$ space for finding the largest cardinality matching in the insert-delete model where $k$ is an upper bound on the largest matching size.

[Chitnis et al. SODA 16]
Small Matching

Theorem
Suppose $\text{match}(G) \leq k$. There exists a $O(k^2 \log k)$ space algorithm in the insert-delete model that finds the size of the largest matching.

Algorithm:
- Let $c : [n] \rightarrow [b]$ be a 2-wise hash function where $b = 1000k$.
- For each pair $i, j \in [b]$, recover an edge $\{x, y\}$ (if one exists) with $\{c(x), c(y)\} = \{i, j\}$
- Repeat $O(\log k)$ times in parallel and return the largest matching amongst the recovered edges.
When a subgraph has the same size matching...

Lemma

Let $\text{match}(G) \leq k$. Let $U = \{u : \deg_G(u) \geq 10k\}$ and $F$ be edges in $G[V \setminus U]$. Then a subgraph $G'$ satisfies $\text{match}(G') = \text{match}(G)$ if $F \subseteq G'$ and $\deg_{G'}(u) \geq 5k$ for all $u \in U$.

Proof.

- $|U| \leq 2k$ since the min vertex cover has size $\leq 2 \text{match}(G) \leq 2k$ and every node in $U$ must be in the min vertex cover.
- $G'$ contains a matching of size

$$\text{match}(F) + |U|$$

since even after we pick the largest matching in $F$, every and then still be able to match every vertex in $U$.

- This follows because the optimum matching in $F$ "consumes" at most $2k$ potential endpoints, since $\text{match}(G) \leq k$. Hence, each of the (at most $2k$) vertices in $U$ can still be matched to $3k$ possible vertices.
\[ \mathbb{P}[F \subseteq G' \text{ and } \deg_{G'}(u) \geq 5k \; \forall u \in U] \geq 1 - \frac{1}{\text{poly}(k)}. \]

- Let \( c \) be 2-wise independent hash function and \( H \) be a graph with one edge \( \{x, y\} \) with \( c(x) = i, c(y) = j \) for each \( i, j \in [b] \).

**Claim**

For \( e \in F \), \( \mathbb{P}[e \in H] \geq 1/2. \)

**Claim**

For \( u \in U \), \( \mathbb{P}[\deg_H(u) \geq 5k] \geq 1/2 \)

- Repeat \( r = \Theta(\log k) \) times, to boost probabilities to \( 1 - \frac{1}{\text{poly}(k)} \).
- Union bound over \( |F| = O(k^2) \) edges and \( |U| = O(k) \) implies result.
- The fact \( |F| = O(k^2) \) and \( |U| = O(k) \) follows from the promise \( \text{match}(G) \leq k \): the induced graph on \( V \setminus U \) has a matching of size \( \Omega(|F|/k) \) as the maximum degree is \( O(k) \) and this size is at most \( k \). Since all vertices in \( U \) must be in the minimum vertex cover, \( |U| \leq 2k \).
Claim 1: $\Pr[e \in H] \geq 1/2$ for $e \in F$

- Let $C$ be a min vertex cover of $G$ and note $|C| \leq 2k$ because the endpoints of the edges in a maximum matching form a vertex cover.
- Let $e = \{x, y\}$ and consider $A = (C \cup \Gamma(x) \cup \Gamma(y)) \setminus \{x, y\}$
- Then $G[V \setminus A]$ consists of the unique edge $e$.
- If no vertices in $A$ receive hash values equal to $c(x)$ and $c(y)$, then $e$ is unique edge with its pair of hash values and hence is in $H$.
- Since $b = 1000k$ and $|A| \leq 2k + 10k + 10k = 22k$,

$$\Pr[e \in H] \geq 1 - \Pr[\exists a \in A : c(a) = c(x)] - \Pr[\exists a \in A : c(a) = c(y)] \geq 1 - 2|A|/b > 1/2.$$
Claim 2: \( \mathbb{P} [\deg_H(u) \geq 5k] \geq 1/2 \) for \( u \in U \)

- Let \( A = C \setminus \{u\} \). Then \( G[V \setminus A] \) is star with center \( u \) and \( \geq 9k \) leaves. Let \( N = \{v_1, \ldots, v_{9k}\} \) be arbitrary set of \( 9k \) such leaves.
- Let \( X_i = 1 \) if \( v_i \) has the same hash value as some other vertex in \( N \) or a vertex in \( A \). Let \( X = \sum X_i \).
- If \( c(u) \not\in c(A) \) and \( X \leq 4k \), then \( H \) has \( \geq 5k \) edges incident to \( u \).
- This happens with probability at least \( 1/2 \) since

\[
\mathbb{P} [c(u) \in c(A)] \leq \frac{2k}{b} = \frac{1}{500} ,
\]

and

\[
\mathbb{E} [X_i] \leq \frac{|A \cup N| - 1}{b} < \frac{2k + 9k}{b} \leq \frac{1}{50} ,
\]

and so

\[
\mathbb{P} [X \geq 4k] \leq \frac{\mathbb{E} [X]}{4k} \leq \frac{9k \times 1/50}{4k} < \frac{1}{20} .
\]