CMPSCI 711: More Advanced Algorithms

Graphs 5: Planar Matchings

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Overview:

- Introduces planar and low arboricity graphs, Vizing's Theorem.
- An $(\alpha + 2)(1 + \epsilon)$ approx for max matching size in a graph of arboricity α using $O(\epsilon^{-2}\alpha \operatorname{polylog} n)$ space in insert-only model.

  [Cormode et al. ESA 17, McGregor, Vorotnikova SOSA 18]
- Discussion of $(\alpha + 2)(1 + \epsilon)$ approx using $O(\epsilon^{-2}\alpha n^{4/5} \operatorname{polylog} n)$ space in the insert-delete model.

  [Chitnis et al. SODA 16]

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Planar Matching

Definition

A graph has arboricity α is the induced graph on any subset U of nodes has strictly less than $\alpha|U|$ edges. A planar graph, i.e., a graph that can be dawn on the plane such that no two edges intersect, has arboricity 3.

Theorem

Suppose G has arboricity at most α. There exists a $O(\epsilon^{-1} \operatorname{polylog} n)$ space algorithm in the insert-only model that approximates the size of the largest matching up to a factor $2 + \alpha + \epsilon$.

Good Edges

Definition

Say an edge $\{u, v\}$ in the stream is good if the number of edges incident to u or v that appear in the stream after $\{u, v\}$ are both at most α.

Theorem

$match(G) \leq g \leq (\alpha + 2)match(G)$ where g is the number of good edges.

The second inequality follows from Vizing's Theorem since the set of good edges has maximum degree $\alpha + 1$.

Fact (Vizing's Theorem)

Any graph has a matching of size at least $\frac{\text{number of edges}}{1+\text{maximum degree}}$.

Proof that $match(G) \leq g$:

- Let L_u be last $\alpha + 1$ edges incident to node u. Note $\{u, v\} \in L_u \cap L_v$ iff it's good. Say $\{u, v\}$ is wasted if $\{u, v\} \in (L_u \cup L_v) \setminus (L_u \cap L_v)$.
- Let H be the set of nodes with degree at least $\alpha + 1$ and

$$w = \text{number of good edges with exactly no end points in } H$$
$$x = \text{number of good edges with exactly one end point in } H$$
$$y = \text{number of good edges with two end points in } H$$
$$z = \text{number of wasted edges with two end points in } H$$

  and note that $g = w + x + y$.
- Then, $x + 2y + z = \sum_{u \in H} |L_u| = (\alpha + 1)|H|$
- Since the graph has arboricity α, $z + y \leq \alpha|H|$.
- Therefore

$$x + y = (\alpha + 1)|H| - y - z \geq (\alpha + 1)|H| - \alpha|H| = |H| \ .$$

- Let E_L be with no endpoints in H and note that $w = |E_L|$.
- Because at most one edge incident to a vertex in H can appear in the optimal matching,

$$g = w + x + y \geq |E_L| + |H| \geq match(G)$$

Insert-Only Algorithm for Low Arboricity Graphs

Algorithm for estimating g:

1. For $p \in \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \frac{1}{2^{\lfloor \log_2 m \rfloor}}\}$, sample edges w/p p and maintain:

   $$S_p = \{e \text{ sampled and } \leq \alpha \text{ edges on endpoints observed since } e \text{ sampled}\}$$

   and cancel instantiation for p if $|S_p| \geq 12\alpha\tau$ where $\tau = 10\epsilon^{-2} \log n$
2. Return $|S_p|/p$ for largest value of p that isn't cancelled.

Analysis: Result follows by:

- Claim: If $p \leq 2\tau/g$, then instantiation won't be cancelled w.h.p.
- Claim: If $p \geq \tau/g$, then w.h.p.

$$|S_p|/p = (1 \pm \epsilon)g \ .$$

If $p \leq 2\tau/g$, instantiation won't be cancelled w.h.p.

- Let g_t be the number of good edges if we only consider the first t elements of the stream. Note $g_t \leq (\alpha+2)match(G) \leq (\alpha+2)g$
- Let $X = |S_p|$ at time t.
- Then,

$$\mathbb{E}[X] = g_t p \leq (\alpha+2)gp \leq 6\alpha\tau$$

and by the Chernoff Bound,

$$\mathbb{P}[X \geq 36\alpha\tau] \leq 2^{-36\alpha\tau} \leq \frac{1}{\text{poly}(n)}$$

If $p \geq \tau/g$, $|S_p|/p = (1 \pm \epsilon)g$ w.h.p.

- Let $X = |S_p|$ at the end of the stream.
- Then,

$$\mathbb{E}[X] = gp \geq \tau$$

and

$$\begin{aligned}\mathbb{P}[|X - \mathbb{E}[X]| \geq \epsilon\mathbb{E}[X]] &\leq 2\exp(-\epsilon^2 gp/3) \\ &\leq 2\exp(-\epsilon^2 \tau/3) \\ &\leq \frac{1}{\text{poly}(n)}\end{aligned}$$

Basic Idea for Insert-Delete Algorithm

Fact

Let G be a graph with arboricity α. For each edge $e = \{u, v\}$, define

$$w_e = \frac{1}{\max(\deg(u), \deg(v)) + 1}$$

and let $W = \sum_{e \in E} w_e$. Then $match(G)/(\alpha + 2) \leq W \leq match(G)$.

Algorithm:

- If $match(G) \leq n^{2/5}$, we can find it exactly in $O(n^{4/5} \log n)$ space using the algorithm for finding "small" matchings.
- If $match(G) \geq n^{2/5}$, estimate W by sampling $O(\epsilon^{-2} n^{4/5} \log n)$ nodes U uniformly at random and compute $\sum_{e \in G[U]} w_e$ to estimate W.