Overview:

- A $2 + \epsilon$ approx for the max weighted matching in $O(\epsilon^{-1} n \log n)$ space.

[Paz and Schwartman. SODA 17]
Definition
A matching in graph $G = (V, E)$ is a subset of edges $M \subset E$ such that no two edges share an end point.

Problem
Find a matching $M$ that maximizes $|M|$. If edges are weighted, we want to maximize $w(M) = \sum_{e \in M} w(e)$.

We’ll assume all weights are $1, 2, \ldots, \text{poly}(n)$. 
Unweighted Matching

- Let $M \leftarrow \emptyset$
- For each new edge $e$: add $e$ to $M$ if no edges in $M$ share an endpoint with $e$

Theorem

*Algorithm uses $O(n \log n)$ space and returns a $2$ approximation to the maximum weighted matching.*

Proof.

- Let $\text{OPT} = \{o_1, o_2, \ldots\}$ be set of edges in the optimal solution.
- Let $M$ be final set of selected edges and note $M$ is maximal.
- For each $e \in \text{OPT}$, charge $1$ to an edge $f \in M$ that shares an endpoint of $e$. There must exist such $f$ because $M$ is maximal.
  Every edge in $M$ gets charged as most $2$. Hence,

\[ 2|M| \geq \text{charges received} = \text{charges made} = |\text{OPT}| \]
Weighted Matching Algorithm

- $H \leftarrow \emptyset$ and $\phi(v) \leftarrow 0$ for all $v \in V$
- For each $e_i = \{u, v\} \in \{e_1, e_2, \ldots, e_m\}$:
  - If $w(e_i) > (1 + \epsilon)(\phi(u) + \phi(v))$:
    - $H \leftarrow H \cup \{e_i\}$
    - $\phi(u) \leftarrow \phi(u) + (w(e_i) - \phi(u) - \phi(v))$
    - $\phi(v) \leftarrow \phi(v) + (w(e_i) - \phi(u) - \phi(v))$
- Construct a greedy matching in $H$ by considering edges in the reverse order they were added.

Lemma

$|H| = O(\epsilon^{-1} n \log n)$

Lemma

Algorithm is a 2 approximation when $\epsilon = 0$.

Corollary

Algorithm is a $2(1 + \epsilon)$ approximation.
Algorithm stores at most $O(\epsilon^{-1} n \log n)$ edges.

- Consider an arbitrary vertex $v$ in the graph.
- The value of $\phi(v)$ is set to at least 1 when the first edge incident to $v$ is added to $H$. Every time another edge that is incident to $v$ is added to $H$, the value of $\phi(v)$ increases to at least

$$\phi(v) + (w(e_i) - \phi(u) - \phi(v)) > \phi(v) + \epsilon(\phi(u) + \phi(v)) \geq (1 + \epsilon)\phi(v).$$

- If $\phi(v)$ becomes larger than the max edge weight, no more edges incident to $v$ are added to $H$.
- At most $\log_{1+\epsilon} \text{poly}(n)$ edges incident to $v$ are added to $H$. 


Algorithm returns a 2 approximation if $\epsilon = 0$: Part 1

- Let max weight matching have edges $M^*$. Let $M$ be the matching returned and define $M_i = M \cap \{e_i, \ldots, e_m\}$. Define edge weights

$$w_i(e) = w(e) - \phi_i(u) - \phi_i(v)$$

where $\phi_i(\cdot)$ are the values just before $i$th edge in the stream. Note

$$w_{i+1}(e) = \begin{cases} 
  w_i(e) & \text{if } e \text{ doesn’t share endpoint with } e_i \\
  w_i(e) - w_i(e_i) & \text{if } e \text{ shares one endpoint with } e_i \\
  w_i(e) - 2w_i(e_i) & \text{if } e = e_i 
\end{cases}$$

if $e_i$ was added to $H$ and $w_{i+1} = w_i$ otherwise.

- Will show $w_i(M^*) \leq 2w_i(M_i)$ for all $i$ by induction on decreasing $i$.

- **Base case:** $w_m(M^*) \leq 2w_m(M_m)$ because all $w_m(e) \leq 0$ for all edges except possibly $e_m$.

- **Induction hypothesis:** $w_{i+1}(M^*) \leq 2w_{i+1}(M_{i+1})$. 

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Algorithm returns a 2 approximation if $\epsilon = 0$: Part 2

- If $e_i \not\in H$ then $w_i = w_{i+1}$ and so
  \[
  w_i(M^*) = w_{i+1}(M^*) \leq 2w_{i+1}(M_{i+1}) = 2w_i(M_{i+1}) = 2w_i(M_i)
  \]

- Otherwise, assume $e_i \in H$ and let $N$ be set of edges intersecting $e_i$ in $G$. Then,
  \[
  w_i(M^*) \leq w_{i+1}(M^*) + 2w_i(e_i) \leq 2w_{i+1}(M_{i+1}) + 2w_i(e_i)
  \]
  since at most two edges are in $N \cap M^*$ and other weights stay same.

- Since $M_i$ has at least one edge in $N$ and hence
  \[
  w_i(M_i) \geq w_{i+1}(M_i) + w_i(e_i) \geq w_{i+1}(M_{i+1}) + w_i(e_i)
  \]
  and therefore $w_i(M^*) \leq 2w_i(M_i)$ as required.
Algorithm returns a $2(1 + \epsilon)$ approximation

Define a new set of edge weights $w'$ as follows: Run the algorithm with $\epsilon > 0$ and when we encounter $e$, define

$$w'(e) = \begin{cases} \frac{w(e)}{1 + \epsilon} & \text{if } \phi(u) + \phi(v) < w(e) \leq (1 + \epsilon)(\phi(u) + \phi(v)) \\ w(e) & \text{otherwise} \end{cases}$$

- Running algorithm with rule “add to $H$ if $w'(e) > \phi(u) + \phi(v)$” is same as using rule “add to $H$ if $w(e) > (1 + \epsilon)(\phi(u) + \phi(v))$”
- We know using the first rule finds matching $M$ with

$$w'(M) \geq w'(M_w^*)/2.$$ 

where $M_w^*$ is the edges in the optimal matching with respect to $w$.

- Since $w(\cdot)/(1 + \epsilon) \leq w'(\cdot) \leq w(\cdot)$,

$$w(M) \geq w'(M) \geq w'(M_w^*)/2 \geq w'(M_w^*)/2 \geq \frac{w(M_w^*)}{2(1 + \epsilon)}$$ 

where $M_w^*$ is the edges in the optimal matching with respect to $w$. 