CMPSCI 711: More Advanced Algorithms

Section 5-2: Information Statistics

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Information Statistics Approach

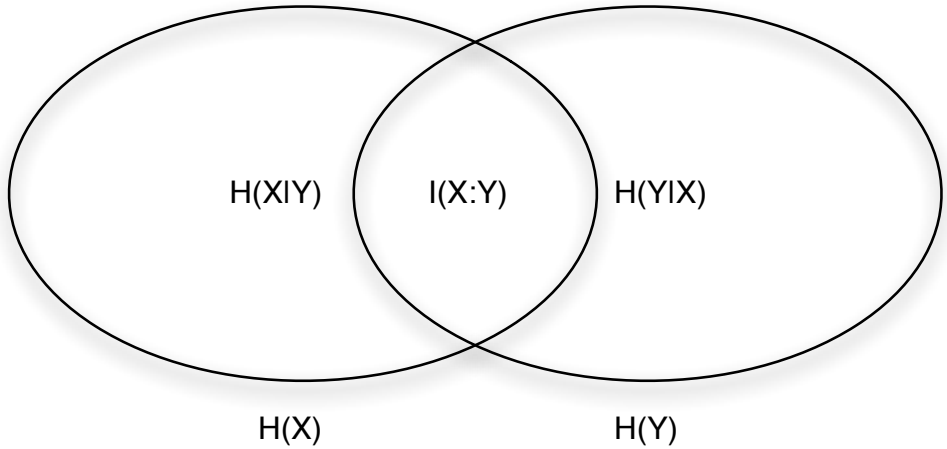
- Information statistics approach is based on analyzing the "information revealed" about the input from the messages.
- Useful for proving bounds on complicated functions in terms of simpler problems, e.g., proving a bound on

$$\textsc{disj}_t(M) = \bigvee_{j \in [n]} \textsc{and}_t(M_{1,j}, \ldots, M_{t,j})$$

  by first establishing a bound on $\textsc{and}_t$.
- We'll first give some definitions and then run through an example.

Information Theory Definitions

- Let X and Y be random variables.
- *Entropy:* $H(X) := \sum_i -\mathbb{P}[X = i] \lg \mathbb{P}[X = i]$
- *Conditional Entropy:* $H(X|Y) := \mathbb{E}_{y \sim Y}[H(X|Y = y)] \leq H(X)$
- *Mutual Information:* $I(X : Y) = H(X) - H(X|Y)$



- Useful Facts:
  - If X takes at most 2^ℓ values, then $H(X) \leq \ell$.
  - Chain rule: $H(XY) = H(X) + H(Y|X)$.
  - Subadditivity: $H(XY) \leq H(X) + H(Y)$; equality if independent.

Mutual Information

Lemma

If X and Y are independent, then $I(XY : Z) \geq I(X : Z) + I(Y : Z)$.

Proof.

$$\begin{aligned} I(XY : Z) &= H(XY) - H(XY|Z) \\ &= H(X) + H(Y) - H(XY|Z) \\ &\geq H(X) + H(Y) - H(X|Z) - H(Y|Z) \\ &= I(X : Z) + I(Y : Z) \end{aligned}$$

□

Information Cost

- Suppose you have a protocol Π for a two-party communication problem P in which Alice and Bob have random inputs X and Y.
- Let M be the (random) message sent by Alice and define:

$$\text{cost}(\Pi) = \max |M|$$

and

$$\text{icost}(\Pi) = I(M : X)$$

- Note $\text{icost}(\Pi) = I(M : X) \leq H(M) \leq \text{cost}(\Pi)$.

Example: Indexing

- We'll prove a lower bound on the information cost of $\textsc{Index}$ where $X \in_R \{0,1\}^n$ in terms a simpler problem "$\textsc{Echo}$"
- $\textsc{Echo}$: Alice has a single bit $B \in_R \{0,1\}$ and Bob wants to output B with probability at least $1 - \delta$.
- A protocol $\Pi_{\textsc{Index}}$ for $\textsc{Index}$ yields a protocol $\Pi_{\textsc{Echo},i}$ for $\textsc{Echo}$:
  1. Given B, Alice picks $X_j \in_R \{0,1\}$ for $j \neq i$ and generates:

$$X = (X_1, X_2, \ldots, X_{i-1}, B, X_{i+1}, \ldots, X_n)$$

  2. She sends the message M she'd have sent in $\Pi_{\textsc{Index}}$ if she'd had X.
  3. Bob receives message and outputs the value he'd have returned in $\Pi_{\textsc{Index}}$ had his input been i.

Relating Information Cost of INDEX and ECHO

- Since $X_1, X_2, \ldots, X_n$ are independent:

$$\begin{aligned}
\text{cost}(\Pi_{\text{INDEX}}) &\geq \text{icost}(\Pi_{\text{INDEX}}) \\
&= I(X_1 X_2 \ldots X_n : M) \\
&\geq I(X_1 : M) + I(X_2 : M) + \ldots + I(X_n : M) \\
&= \text{icost}(\Pi_{\text{ECHO},1}) + \text{icost}(\Pi_{\text{ECHO},2}) + \ldots + \text{icost}(\Pi_{\text{ECHO},n})
\end{aligned}$$

- *Lemma:* Any protocol solving ECHO with probability $\geq 1 - \delta$, needs

$$\text{icost}(\Pi_{\text{ECHO},i}) \geq 1 - H_2(\delta)$$

  where $H_2(p) = -p \lg p - (1-p) \lg(1-p)$.
- Hence, $\text{cost}(\Pi_{\text{INDEX}}) \geq (1 - H_2(\delta))n$.

Proof of Lemma

1. Fano's inequality: Let A and B be random variables. If you can guess B correctly with probability at least $1 - \delta$ given A, then

$$H(B|A) \leq H_2(\delta) \ .$$

2. Let $A = M$ be message and B be the bit needing echoed.
3. Hence,

$$\text{icost}(\Pi_{\text{ECHO}}) = H(B) - H(B|M) \geq 1 - H_2(\delta)$$

Outline for DISJ_t Lower Bound

- Express DISJ_t in terms of AND_t where $\text{AND}_t(x_1, \ldots, x_t) = \prod_i x_i$:

$$\text{DISJ}_t(M) = \bigvee_{j \in [n]} \text{AND}_t(M_{1,j}, \ldots, M_{t,j})$$

- Consider a random input M to DISJ_t where $M_{D_j j} \in_R \{0, 1\}$ for $D_j \in_R [t]$. All other entries are 0.
- Let $T = (T_1, \ldots, T_{t-1})$ be the messages sent in a t-party protocol and define the information cost of a protocol as:

$$\text{icost}(\Pi | D) = I(T : M | D) \quad \text{where} \quad D = (D_1, \ldots, D_t) \ .$$

- A protocol for DISJ_t yields n different protocols $\Pi_{\text{AND}_t, i}$ for AND_t:

$$\text{icost}(\Pi_{\text{DISJ}_t} | D) \geq \sum_{i \in [n]} \text{icost}(\Pi_{\text{AND}_t, i} | D) \ .$$

- Result follows by showing $\text{icost}(\Pi_{\text{AND}_t, i} | D) = \Omega(1/t)$.