Outline

Probability Amplification with Two Point Sampling

Probability Amplification with Expanding Graphs

Probability Amplification with Random Walks on Expanders

Bonus Section! Connectivity and Eigenvectors

Readings
Consider language $L$ and randomized algorithm $A$ such that for random $r \in \{0, \ldots, p - 1\}$ ($p$ is prime),

- If $x \in L$, then $\mathbb{P}[A(x, r) = 1] \geq 1/2$
- If $x \not\in L$, then $\mathbb{P}[A(x, r) = 0] = 1$
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If algorithm has worst case polynomial running time in $|x|$, we call it an $RP$ algorithm.
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Easy to boost the probability of returning 1 for $x \in L$ up to $1 - 2^{-t}$ using $O(t \log p)$ random bits.
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- Easy to boost the probability of returning 1 for $x \in L$ up to $1 - 2^{-t}$ using $O(t \log p)$ random bits.
- What if we only have $O(\log p)$ bits?
Two Point Sampling

- Let $p$ be a prime and let $a, b$ be chosen uniformly at random from $\{0, 1, \ldots, p-1\}$.
- Define $r_0, \ldots, r_{t-1}$ where $r_i = ai + b \pmod{p}$.
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Let $Y = \sum_{i=1}^{t} A(x, r_i)$
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- **Exercise:** $r_i$ are pairwise independent (but not three-wise)
- Let $Y = \sum_{i=1}^{t} A(x, r_i)$
- **Exercise:** If $x \in L$, $\mathbb{P}[Y = 0] \leq 1/t$
- Hence, with $O(\log p)$ random bits, can boost probability to $1 - 1/t$ given $t$ trials.
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Readings
We’ll show approach with error probability $\approx 1/n^{\log n}$ using $\log^2 n$ random bits (where there are $O(n)$ witnesses).
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Approach is based on “sparse expanding graphs”
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Approach is based on “sparse expanding graphs”

**Theorem**

For sufficiently large $n$, there is a bipartite graph $G = (L, R, E)$ with $|L| = n$, $|R| = 2^{\log^2 n}$ with:

1. Every subset of $n/2$ vertices from $L$ has at least $2^{\log^2 n} - n$ neighbors in $R$.
2. No vertex of $R$ has more than $12 \log^2 n$ neighbors.
Proof of Theorem

- By probabilistic method: For each vertex in $L$, pick $d = 2^{\log^2 n}(4 \log^2 n)/n$ neighbors in $R$ (with replacement).

- Probability there is a set of $n/2$ vertices in $L$ with fewer than $2^{\log^2 n} - n$ neighbors in $R$ is at most $(n/2)\left(2^{\log^2 n}\right)\left(2^{\log^2 n} - n\right)^{dn/2} ≪ 1/2$.

- Second condition: expected number of neighbors of $v \in R$ is $4 \log^2 n$.

- Chernoff bound shows that there exists $v \in R$ with more than $12 \log^2 n$ neighbors with probability $\leq |R|\left(e/3\right)^{12 \log^2 n} ≪ 1/2$. 
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- Pick \( v \in_R R \): Uses \( O(\log^2 n) \) random bits.
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- At least \( n/2 \) of nodes in \( L \) are witnesses if \( x \in L \).
- Probability we find a witness is at least \( 1 - \frac{n}{2^{\log^2 n}} \).
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Readings
Expanders

Definition
An \((n, d, c)\)-expander is a \(d\)-regular bipartite (multi-)graph \(G = (X, Y, E)\) with \(|X| = |Y| = n/2\) such that for any \(S \subset X\),

\[
|\Gamma(S)| \geq \left(1 + c \left(1 - \frac{2|S|}{n}\right)\right)|S|.
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**Example (Gabber-Galil expanders)**
For positive integer \(m\), let \(n = 2m^2\). Each vertex in \(X\) has distinct label consisting of a pair \((a, b)\) for \(a, b \in \{0, \ldots, m - 1\}\). Similarly for \(Y\). \((x, y) \in X\) has neighbors in \(Y\) with labels:

\[
(x, y), (x, 2x + y), (x, 2x + y + 1), (x, 2x + y + 2)
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(x + 2y, y), (x + 2y + 1, y), (x + 2y + 2, y)
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where addition is modulo \(m\). Resulting graph is a \((n, 7, 2\alpha)\) example where \(\alpha = (2 - \sqrt{3})/4\).
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Eigenvalues and Algebraic Graph Theory

Fact

For symmetric $n \times n$ matrix $A$, there exists $n$ orthonormal eigenvectors $e_1, \ldots, e_n$ with eigenvectors $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$: i.e.,

$$e_i A = \lambda_i e_i \quad \text{and} \quad e_i \cdot e_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

If $A$ is adjacency matrix of graph $G$, then $G$ connected implies $\lambda_2 < \lambda_1$. If $G$ is $d$-regular and bipartite, $\lambda_1 = d$ and $\lambda_n = -d$. 
Eigenvalues and Algebraic Graph Theory

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Theorem (Alon '86)
If $G$ is an $(n, d, c)$-expander then $|\lambda_2| \leq d - \frac{c^2}{1024 + 2c^2}$. If $|\lambda_2| \leq d - \epsilon$ then $G$ is an $(n, d, c)$-expander with $c \geq \frac{2d\epsilon - \epsilon^2}{d^2}$. 
Expander are Rapidly Mixing

Definition
Let $G$ be an $(n, d, c)$-expander and let $Q$ be transition matrix on $G$ with $Q_{uu} = 1/2$ and $Q_{uv} = 1/(2d)$ if $v$ is a neighbor of $u$. Let $\lambda_i$ be eigenvalues of $G$ and $\mu_i = (1 + \lambda_i/d)/2$ be eigenvalues of $Q$. 
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Theorem
Consider aperiodic random walk on $G$ defined by $Q$. Distribution at time $t$,

$$|q^{(t)} - \pi| \leq 2\sqrt{n}\mu_2^t$$

This is at most $\sqrt{n}(1 - \epsilon/2d)^t$ if $\lambda_2 \leq d - \epsilon$. 

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Lemma
For any vector $v \in \mathbb{R}^n$, $\|v\|_2 \leq \|v\|_1 \leq \sqrt{n}\|v\|_2$ where

$$\|v\|_1 = \sum_i |v_i| \text{ and } \|v\|_2 = \sqrt{\sum_i v_i^2}.$$
Proof of Theorem

- Distribution at time $t$ is $q(t) = q(0)Q^t$
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- Express $q(0)$ as $q(0) = \sum_{i \in [n]} c_i e_i$
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- Let $x = c_1 e_1$ and using previous lemma:

$$\|q(t) - x\|_1 \leq \sqrt{n} \|q(t) - x\|_2 = \sqrt{n} \sum_{i=2}^n c_i e_i \mu_i^t \|_2$$
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- Because $\mu_2 \geq \mu_3 \geq \ldots \mu_n \geq 0$,

$$\|q^{(t)} - x\|_1 \leq \sqrt{n} \mu_2^t \| \sum_{i=2}^n c_i e_i \|_2 \leq \sqrt{n} \mu_2^t$$
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If algorithm has worst case polynomial running time in $|x|$, we call it an $\textit{BPP}$ algorithm.
Let $G$ be a $(N, 7, 2\alpha)$ Gabber-Galil expander where $N = 2^n$ and nodes are labelled by $\{0, 1\}^n$. 

Consider random walk on $G$ with prob. $1/2$ of not moving in a given step and random starting position: $X_0, X_1, X_2, \ldots, X_{7^{\beta}}$ where $\beta = O(1)$ satisfies $\lambda^{\beta/2} \leq 1/10$.

Total random bits requires $n + O(k)$. 

For $0 \leq i \leq 7^\beta$, let $r_i$ be label of $X_i$. 

**Theorem** 

Majority of $A(x, r_0), \ldots, A(x, r_{7^\beta})$ are correct with prob. $1 - 1/2^k$. 


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Proof of Theorem

Let $W \in \{0, 1\}^{N \times N}$ with $W_{uu} = 1$ iff node $u$ is “good”
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- Let $B = Q^t$. For $A \in [7k]$, probability $r_i$'s are good for $i \in A$ and $r_i$'s are not good for $i \not\in A$ is:

$$\|q^{(0)} BS_1 \ldots BS_{7k}\|_1$$

where $S_i = W$ if $i \in A$ and $S_i = \bar{W} = I - W$ if $i \not\in A$. 

Probability $|A| \leq 7k/2$ is at most $2^{7k}(1/5)^{7k/2} < 1/2^k$.
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Claim

For any $p \in \mathbb{R}^N$, $\|pBW\|_2 \leq \|p\|_2$ and $\|pB\bar{W}\|_2 \leq \|p\|_2/5$
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Claim

*For any $p \in \mathbb{R}^N$, $\| p B W \|_2 \leq \| p \|_2$ and $\| p B \bar{W} \|_2 \leq \| p \|_2 / 5$*

- Applying claim repeatedly,

\[ \| q^{(0)} B S_1 \ldots B S_{7k} \|_1 \leq \sqrt{N} \| q^{(0)} B S_1 \ldots B S_{7k} \|_2 \leq \sqrt{N} (1/5)^{7k - |A|} \| q^{(0)} \|_2 = (1/5)^{7k - |A|} \]
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$$\|q^{(0)} BS_1 \ldots BS_{7k}\|_1 \leq \sqrt{N}\|q^{(0)} BS_1 \ldots BS_{7k}\|_2 \leq \sqrt{N}(1/5)^{7k-|A|}\|q^{(0)}\|_2 = (1/5)^{7k-|A|}$$

- Probability $|A| \leq 7k/2$ is at most $2^{7k}(1/5)^{7k/2} < 1/2^k$
Finishing the Theorem: Proof of Claim

- Express \( p \) in eigenvector basis \( p = \sum_{i \in [n]} c_i e_i \)
Finishing the Theorem: Proof of Claim

Express $p$ in eigenvector basis $p = \sum_{i \in [n]} c_i e_i$

Because eigenvectors are in $[0, 1]$:

$$\|p B W\|_2 \leq \|p B\|_2 = \left\| \sum_{i \in [n]} c_i \lambda_i^\beta e_i \right\|_2 = \sqrt{\sum_{i \in [n]} c_i^2 \lambda_i^{2\beta}} \leq \sqrt{\sum_{i \in [n]} c_i^2}$$
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- Write $p = c_1 e_1 + y$ for $y = \sum_{i \geq 2} c_i e_i$. $\lambda_2^\beta \leq 1/10$ implies:

$$\|yB\bar{W}\|_2 \leq \|yB\|_2 = \| \sum_{i \geq 2} c_i \lambda_i^\beta e_i \|_2 \leq \lambda_2^\beta \sqrt{\sum_{i \geq 2} c_i^2} \leq \|y\|_2/10$$
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- Since $e_1 = (1, \ldots, 1)/\sqrt{n}$ and $\lambda_1 = 1$:  
  \[ \|c_1 e_1 B\bar{W}\|_2 \leq \|c_1 e_1 \hat{W}\|_2 \leq \|c_1 e_1\|_2 / 10 \]
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  \]
- Write $p = c_1 e_1 + y$ for $y = \sum_{i\geq 2} c_i e_i$. $\lambda_2^\beta \leq 1/10$ implies:
  \[
  \|y B \tilde{W}\|_2 \leq \|y B\|_2 = \| \sum_{i\geq 2} c_i \lambda_i^\beta e_i \|_2 \leq \lambda_2^\beta \sqrt{\sum_{i\geq 2} c_i^2} \leq \|y\|_2/10
  \]
- Since $e_1 = (1, \ldots, 1)/\sqrt{n}$ and $\lambda_1 = 1$:
  \[
  \|c_1 e_1 B \tilde{W}\|_2 \leq \|c_1 e_1 \hat{W}\|_2 \leq \|c_1 e_1\|_2/10
  \]
- Putting it together:
  \[
  \|p B \tilde{W}\|_2 \leq \|c_1 e_1 B \tilde{W}\|_2 + \|y B \tilde{W}\|_2 \leq (\|c_1 e_1\|_2 + \|y\|_2) \leq \|p\|_2/5
  \]
Outline

Probability Amplification with Two Point Sampling

Probability Amplification with Expanding Graphs

Probability Amplification with Random Walks on Expanders

Bonus Section! Connectivity and Eigenvectors

Readings
Connectivity and Eigenvectors

Theorem

Let $A$ be the adjacency matrix of a $d$-regular graph. Then a) $\lambda_1 = d$, and b) $\lambda_2 = d$ iff graph is disconnected.

Proof.
Connectivity and Eigenvectors

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Proof.

Part a): Let $x \in \mathbb{R}^n$ satisfy $\|x\|_2 = 1$ and $xA = \lambda_1 x$:

$$0 \leq \sum_{u,v} A_{u,v} (x(u) - x(v))^2 = 2d \sum_v x(v)^2 - 2 \sum_{u,v} x(u)x(v)A_{u,v}$$

$$= 2d - 2xAx^T = 2d - 2\lambda_1$$
Connectivity and Eigenvectors

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= 2d - 2xAx^T = 2d - 2\lambda_1
\]

▶ Hence $d \geq \lambda_1$ and therefore $d = \lambda_1$ since $\lambda_1 \geq d$
Connectivity and Eigenvectors

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  $= 2d - 2x Ax^T = 2d - 2\lambda_1$

  Hence $d \geq \lambda_1$ and therefore $d = \lambda_1$ since $\lambda_1 \geq d$

- **Part b):** Let $\lambda_2 = d$ and $e_2 \perp e_1 = (1, \ldots, 1)/\sqrt{n}$:

  $0 \leq \sum_{u,v} A_{u,v} (e_2(u) - e_2(v))^2 = 2d - 2\lambda_2$
Connectivity and Eigenvectors

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Let $A$ be the adjacency matrix of a $d$-regular graph. Then a) $\lambda_1 = d$, and b) $\lambda_2 = d$ iff graph is disconnected.

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  Hence $d \geq \lambda_1$ and therefore $d = \lambda_1$ since $\lambda_1 \geq d$

- Part b): Let $\lambda_2 = d$ and $e_2 \perp e_1 = (1, \ldots, 1)/\sqrt{n}$:

  
  \[
  0 \leq \sum_{u,v} A_{u,v} (e_2(u) - e_2(v))^2 = 2d - 2\lambda_2
  \]

  $e_2(u) = e_2(v)$ for all $u, v$ if graph connected so $e_2 \not\perp e_1$
Proof Continued

- Part b) other direction: Suppose \( G \) is disconnected and \( S, V \setminus S \) is partition of graph.
Proof Continued

- Part b) other direction: Suppose $G$ is disconnected and $S, V \setminus S$ is partition of graph.
- Let $p = \frac{|S|}{|V|}$ and $q = \frac{|V \setminus S|}{|V|}$ and define

$$x(v) = \begin{cases} 
q & \text{if } v \in S \\
-p & \text{if } v \notin S 
\end{cases}$$
Proof Continued

- Part b) other direction: Suppose $G$ is disconnected and $S, V \setminus S$ is partition of graph.
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$$x(v) = \begin{cases} 
q & \text{if } v \in S \\
-p & \text{if } v \notin S 
\end{cases}$$

- $x \perp e_1$ since

$$x \cdot e_1 = n^{-0.5} \sum_{v} x(v) = n^{-0.5} (q|S| - p|V \setminus S|) = n^{-0.5} (qpn - pqn) = 0$$
Part b) other direction: Suppose $G$ is disconnected and $S, V \setminus S$ is partition of graph.

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$x \perp e_1$ since

$$x \cdot e_1 = n^{-0.5} \sum_v x(v) = n^{-0.5} (q|S| - p|V \setminus S|) = n^{-0.5} (qpn - pqn) = 0$$

But $x$ also has eigenvalue $d$: $xM = dx$
Outline

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Bonus Section! Connectivity and Eigenvectors

Readings
Readings

For next time, please make sure you’ve read:

- Chapter 3.4, 5.3 [MR].
- Chapter 6 [MR] and 11.1, 11.2 from [MU]