CMPSCI 711: "Really Advanced Algorithms"

Lecture 8 – Probabilistic Method and Lovász Local Lemma

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Outline

Probabilistic Method

Lovász Local Lemma

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Examples we've already seen...

- Any graph has a cut of size at least $m/2$
- For any collection of m clauses, it is possible to satisfy $(1 - 2^{-k})m$ of the clauses if each clause has at least k literals.
- For any collection of subsets $A_1, \ldots, A_n \subset [n]$, it is possible to partition $[n]$ into A and B such that

$$\max_{i \in [n]} \big| |A_i \cap B| - |A_i \cap C| \big| \leq 4\sqrt{n \ln n}$$

k-SAT

- Input: A CNF formula consisting of m clauses in n Boolean variables $x_1, \ldots, x_n$, e.g.,

$$(x_1 \vee x_2 \vee \bar{x}_3) \wedge \ldots \wedge (x_9 \vee x_{10} \vee x_{21})$$

  where $\bar{x}_i = 1 - x_i$.
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Theorem

If each clause contains exactly $k \geq 3$ literals and each variable appears (complemented or un-complemented) in at most $2^{k/50}$ clauses then the formula can always be satisfied.

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- E.g., if $\sum_i \mathbb{P}[B_i] < 1$, we are good.
- What if probabilities aren't tiny and events not independent?

Lovász Local Lemma

Theorem

Consider events $B_1, \ldots, B_n$ *with* B_i *independent of* $\{B_j : j \notin \Gamma(i)\}$*.*
Suppose that there exist $x_i \in [0,1]$ *for* $i \in [n]$ *such that*

$$\mathbb{P}\left[B_i\right] \leq x_i \prod_{j \in \Gamma(i)} (1 - x_j)$$

Then, $\mathbb{P}\left[\cap_{i \in [n]} G_i\right] \geq \prod_{i \in [n]} (1 - x_i)$ *where* $G_i = \bar{B}_i$*.*

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Corollary

Let $B_1, \ldots, B_n$ *be events with* $\mathbb{P}\left[B_i\right] \leq p$ *and* B_i *independent of all but at most* d *other events, then* $\mathbb{P}\left[\cap_{i \in [n]} G_i\right] > 0$ *if* $ep(d+1) \leq 1$*.*

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Consider events $B_1, \ldots, B_n$ with B_i independent of $\{B_j : j \notin \Gamma(i)\}$. Suppose that there exist $x_i \in [0,1]$ for $i \in [n]$ such that

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Corollary

Let $B_1, \ldots, B_n$ be events with $\mathbb{P}\left[B_i\right] \leq p$ and B_i independent of all but at most d other events, then $\mathbb{P}\left[\cap_{i \in [n]} G_i\right] > 0$ if $ep(d+1) \leq 1$.

Proof.

Use Lovász Local Lemma with $x_i = 1/(d+1)$ for all $i \in [n]$. $\square$

Proof of Lovász Local Lemma

- Sufficient to prove $\mathbb{P}[B_i | \cap_{j\in S} G_j] \leq x_i$ for any $S \subset [n], i \notin S$:

  $\mathbb{P}\left[\cap_{i\in[n]} G_i\right] = (1-\mathbb{P}[B_1])(1-\mathbb{P}[B_2|G_1])\ldots(1-\mathbb{P}\left[B_n | \cap_{i\in[n-1]} B_i\right])$

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- Inductive Step: Let $S_1 = \{j \in S : j \in \Gamma(i)\}$ and $S_2 = S \setminus S_1$:

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- Numerator: By independence assumptions

$$\mathbb{P}\left[B_i \cap (\cap_{j \in S_1} G_j) \,\middle|\, \cap_{j \in S_2} G_j\right] \leq \mathbb{P}\left[B_i \middle| \cap_{j \in S_2} G_j\right] = \mathbb{P}[B_i]$$

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- Denominator: Let $S_1 = \{j_1, \ldots, j_r\}$ & $T_k = S_2 \cup \{j_1, \ldots, j_k\}$

$$\begin{aligned} \mathbb{P}[G_{j_1} \cap \ldots \cap G_{j_r} | \cap_{j \in S_2} G_j] &= \prod_{k=0}^{r-1} \left(1 - \mathbb{P}\left[B_{j_{k+1}} | \cap_{j \in T_k} G_j\right]\right) \\ &\ge \prod_{j \in \Gamma(i)} (1 - x_j) \ge \mathbb{P}[B_i] / x_i \end{aligned}$$

Return to k-SAT

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- B_j is independent of all but at most $d = k(2^{k/50} - 1)$ other events.
- Since $ep(d + 1) \leq 1$ for $k \geq 3$, using LLL: $\mathbb{P}\left[\cap_{i \in [n]} G_i\right] > 0$

$\square$