Outline

Clock Solitaire and Principle of Deferred Decisions

Stable Matching Problem

Probabilistic Analysis of Gale-Shapley Algorithm

Readings
Recall Last Week’s Puzzle

- Take a standard pack of 52 cards that is randomly shuffled.
- Split into 13 piles of 4 and label piles \{A,2,\ldots,10,J,Q,K\}.
- Take first card from “K” pile.
- Take next card from “X” pile where X is the face value of the previous card taken.
- Repeat until either all cards are removed (you win) or we get stuck (you lose).

What’s the probability you win?
Structural Observations

Lemma

The last card before we terminate (either winning or loosing) is K.
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Proof.

- Suppose at iteration $j$ we draw card X but pile “X” is empty.
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Proof.

- Suppose at iteration \( j \) we draw card X but pile “X” is empty.
- If pile ‘X” is empty and \( X \neq K \) then we have already drawn 4 copies of card X prior to iteration \( j \). Contradiction!
Structural Observations

Lemma
The last card before we terminate (either winning or loosing) is $K$.

Proof.

- Suppose at iteration $j$ we draw card $X$ but pile “$X$” is empty.
- If pile ‘$X$’ is empty and $X \neq K$ then we have already drawn 4 copies of card $X$ prior to iteration $j$. Contradiction!

Lemma
We win iff the fourth $K$ is the 52nd card.
Structural Observations

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Proof.
- When 1st, 2nd, or 3rd K is seen we don’t terminate because “K” pile is non-empty.
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- Suppose at iteration \( j \) we draw card \( X \) but pile “X” is empty.
- If pile ‘X’ is empty and \( X \neq K \) then we have already drawn 4 copies of card \( X \) prior to iteration \( j \). Contradiction!

Lemma

We win iff the fourth \( K \) is the 52nd card.

Proof.

- When 1st, 2nd, or 3rd \( K \) is seen we don’t terminate because “K” pile is non-empty.
- Terminate when 4th \( K \) is seen: we win iff it’s the 52nd card.
Principle of Deferred Decisions

- How do we compute the probability that the fourth K is the 52nd card? \( P \text{[fourth K is 52nd card]} \) equals:

\[
\frac{\text{\# game configurations such that K is 52nd card revealed}}{\text{\# game configurations}}
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- **Principle of Deferred Decisions:** Let the random choices unfold with the progress of the analysis rather than fixing random events upfront.
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For clock solitaire this means we may assume that at each draw, any unseen card is equally unlikely.
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Theorem

The probability we win clock solitaire is \( 1/13 \).
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Readings
Stable Matching Problem

Consider a society in which there are $n$ women $(w_1, \ldots, w_n)$ and $n$ men $(m_1, \ldots, m_n)$.

- A matching is a 1–1 correspondence between the men and the women (we assume a monogamous, heterosexual society).

Does a stable matching always exist? Can we find one in polynomial time?
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- Each female has a (strict) preference list for the males and each male has a (strict) preference list for the females.
- The matching is unstable if there exists \( w_i \) and \( m_j \) such that
  - \( w_i \) and \( m_j \) are not matched to each other.
  - \( w_i \) prefers \( m_j \) to her match.
  - \( m_j \) prefers \( w_i \) to his match.

A configuration that is not unstable is stable.

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The Gale-Shapley “Proposal” Algorithm

- Let $i$ be the smallest value such that $m_i$ is unmatched.
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- She accepts if either a) she is currently unmatched, or b) she finds $m_i$ more desirable than her current match (in which case her current match becomes unmatched.)

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- Repeat until there are no unmatched men left.
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Algorithm is Well-Defined

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Proof.
- Once a woman becomes matched she doesn’t become unmatched (although she may change her match.)
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**Lemma**

*Whenever there’s an unmatched man* $m_i$, *there is someone he hasn’t proposed to.*

**Proof.**

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- All the women to which $m_i$ proposed are already matched.
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Whenever there’s an unmatched man $m_i$, there is someone he hasn’t proposed to.

Proof.

- Once a woman becomes matched she doesn’t become unmatched (although she may change her match.)
- All the woman to which $m_i$ proposed are already matched.
- If $m_i$ has proposed to everyone, all the women are matched, hence all the men are matched. Contradiction!
Algorithm is Efficient

Theorem

The algorithm terminates after $O(n^2)$ repeats.
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Proof.

- At each stage of the algorithm, let $t_i$ be the number of women to which $m_i$ could still potentially propose.
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*The algorithm terminates after $O(n^2)$ repeats.*

Proof.

- At each stage of the algorithm, let $t_i$ be the number of women to which $m_i$ could still potentially propose.
- At each step $\sum_{i \in [n]} t_i$ decreases by 1.
Algorithm is Efficient

Theorem
*The algorithm terminates after $O(n^2)$ repeats.*

Proof.

- At each stage of the algorithm, let $t_i$ be the number of women to which $m_i$ could still potentially propose.
- At each step $\sum_{i \in [n]} t_i$ decreases by 1.
- Initially $\sum_{i \in [n]} t_i = n^2$ so there can be at most $n^2$ steps.
Algorithm is Correct

Theorem

The matching found by the Gale-Shapley algorithm is stable.
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Proof.

- Proof by contradiction: Suppose matching includes $m_i - w_j$ and $m_k - w_l$ but $m_i$ and $w_l$ prefer to be matched to each other.
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- Proof by contradiction: Suppose matching includes $m_i-w_j$ and $m_k-w_l$ but $m_i$ and $w_l$ prefer to be matched to each other.
- Since $m_i$ prefers $w_l$ to $w_j$, he must have proposed to $w_l$ before he proposed to $w_j$. 

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- Proof by contradiction: Suppose matching includes $m_i-w_j$ and $m_k-w_l$ but $m_i$ and $w_l$ prefer to be matched to each other.
- Since $m_i$ prefers $w_l$ to $w_j$, he must have proposed to $w_l$ before he proposed to $w_j$.
- But then, $w_l$ must prefer her current match to $m_i$: either she already had a better match when $m_i$ proposed or she matched $m_i$ initially and then got a better proposal. Contradiction!
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Stable Matching Problem

Probabilistic Analysis of Gale-Shapley Algorithm

Readings
In randomized algorithms, we consider algorithms that make random choices and investigate what happens when they process a “fixed input.” E.g., the min-cut algorithm from lecture 1.

Theorem

If the preference lists are random, the expected number of iterations of Gale-Shapley is $\leq nH_n$. 
In randomized algorithms, we consider algorithms that make random choices and investigate what happens when they process a “fixed input.” E.g., the min-cut algorithm from lecture 1.

In probabilistic analysis, we consider random input and investigate what happens when it’s processed by a fixed algorithm. E.g., the Gale-Shapley algorithm when the preference lists are random.
Probabilistic Analysis

- In randomized algorithms, we consider algorithms that make random choices and investigate what happens when they process a “fixed input.” E.g., the min-cut algorithm from lecture 1.

- In probabilistic analysis, we consider random input and investigate what happens when it’s processed by a fixed algorithm. E.g., the Gale-Shapley algorithm when the preference lists are random.

**Theorem**

If the preference lists are random, the expected number of iterations of Gale-Shapley is \( \leq nH_n \).
Principle of deferred decision: we may assume that at each step $m_i$ proposes to a woman chosen uniformly at random from the women that have not yet rejected him.
Probabilistic Analysis of Gale-Shapley Algorithm (1/2)

- **Principle of deferred decision:** we may assume that at each step \( m_i \) proposes to a woman chosen uniformly at random from the women that have no yet rejected him.
- To simplify things, use a modification of the Gale-Shapley algorithm, the “amnesiac” algorithm.
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  - At each step, \( m_i \) proposes to a woman uniformly at random from the set of all \( n \) women.
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  - This doesn’t change the outcome of the algorithm since, if $m_i$ was rejected by $w_j$ before, he’ll be rejected again.
  - The expected running time of the modified algorithm is an upper bound for the running time of original algorithm.
Theorem

If the preference lists are random, the expected number of iterations of Gale-Shapley is at most $nH_n$.

Proof.

Since the algorithm terminates once all women have received at least one proposal, the random process is analogous to the coupon collector problem.
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Readings
For next time, please make sure you’ve read:

- Appendix C: Basic Probability Theory (9 pages)
- Chapter 1: Introduction up to section 1.4 (14 pages)