Outline

Markov Chains

Random Walks on Graphs
Motivating Example

- An algorithm for 2-SAT:
  1. Pick arbitrary assignment.
  2. Pick an unsatisfied clause: randomly flip the value assigned to one of the two variables.
  3. Repeat Step 2 until there are no unsatisfied clauses.
- Let $x(t)$ be the assignment at time $t$.
- Consider $X(t) = n - \Delta(x(t), y)$ for a fixed satisfying assignment $y$, i.e., the number of variables that are set the same in both $x(t)$ and $y$
- $X(t+1) = X(t) \pm 1$ and 
  \[ \mathbb{P} \left[ X(t+1) = X(t) + 1 \right] \geq 1/2 \]
- How long until we terminate?
A Markov chain is a discrete-time stochastic process that defines a sequence of random variables \((X_0, X_1, X_2, \ldots)\) and is defined by:

- **State space:** e.g., \(X_t \in \{1, \ldots, n\}\)
- **Transition probabilities:** \(P_{ij} = \Pr[X_t = j|X_{t-1} = i]\)
- **Initial distribution for** \(X_0\).

**Memoryless:** \(X_t\) only depends on \(X_{t-1}\):

\[
\Pr[X_t = i_t|X_{t-1} = i_{t-1}, \ldots, X_1 = i_1, X_0 = i_0] = \Pr[X_t = i_t|X_{t-1} = i_{t-1}] = P_{i_{t-1}i_t}
\]
State Probability Vector and Stationary Distribution

Definition
Let $q_i^{(t)} = \mathbb{P}[X_t = i]$. The distribution of the chain at time $t$ is the row vector $(q_1^{(t)}, q_2^{(t)}, \ldots, q_n^{(t)})$.

Lemma
$q^{(t)} = q^{(0)} P^t$ where $P$ be the matrix whose $(i, j)$-th entry is $P_{ij}$.

Definition
A stationary distribution for a Markov chain with transition matrix $P$ is a probability distribution $\pi$ such that $\pi = \pi P$. 
Definition
The underlying graph of a Markov chain is a directed graph $G = (V, E)$ where $V = [n]$ and $(i, j) \in E$ iff $P_{ij} > 0$.

Definition
A Markov chain is irreducible if its underlying graph consists of a single strong component. (Recall that a strong component of a directed graph is a maximal subgraph $C$ of $G$ such that for any vertices $i$ and $j$ in $C$, there is a directed path from $i$ to $j$.)
Transient States and Persistent States

Probability that $t$ is the first time that the chain visits state $j$ if it starts at state $i$:

$$r_{ij}^{(t)} = \mathbb{P} [X_t = j \text{ and } X_s \neq j \text{ for all } 1 \leq s < t | X_0 = i]$$

Probability that the chain ever visits state $j$ if it starts at state $i$:

$$f_{ij} = \sum_{t>0} r_{ij}^{(t)}$$

Expected time to until a visit to state $j$ if it starts at state $i$:

$$h_{ij} = \sum_{t>0} tr_{ij}^{(t)} \text{ if } f_{ij} = 1 \text{ and } h_{ij} = \infty \text{ otherwise}$$

**Definition**
If $f_{ii} < 1$ then state $i$ is transient. If $f_{ii} = 1$ then state $i$ is persistent. Those persistent states $i$ for which $h_{ii} = \infty$ are said to be null persistent and those for which $h_{ii} \neq \infty$ are said to be non-null persistent.
Definition
The periodicity of state $i$ is the maximum integer $T$ for which there exists an initial distribution $q^{(0)}$ and positive integer $a$ such that for all $t$ we have: $q_i^{(t)} > 0$ implies $t \in \{a + Ti : i \geq 0\}$. If $T > 1$ then state is periodic and aperiodic otherwise. If every state is aperiodic then the Markov chain is aperiodic.

Definition
An ergodic state is one that is aperiodic and non-null persistent. A Markov chain is ergodic if all states are ergodic.
Fundamental Theorem of Markov Chains

Theorem
Any irreducible, finite, and aperiodic Markov chain has the following properties:

1. All states are ergodic.
2. There is a unique stationary distribution $\pi$ with $\pi_i > 0$.
3. $f_{ii} = 1$ and $h_{ii} = 1/\pi_i$.
4. Let $N(i, t)$ be the number of times the Markov chain visits state $i$ in $t$ steps. Then,
\[
\lim_{t \to \infty} \frac{N(i, t)}{t} = \pi_i
\]
Outline

Markov Chains

Random Walks on Graphs
Random Walks on Graphs

A connected, non-bipartite, undirected graph \( G = (V, E) \) defines Markov Chain \( M_G \) with states \( V \) and transition matrix:

\[
P_{uv} = \begin{cases} 
\frac{1}{d(u)} & \text{if } (u, v) \in E \\
0 & \text{otherwise}
\end{cases}
\]

where \( |V| = n \) and \( |E| = m \).

**Lemma**

\( M_G \) is irreducible & aperiodic because it’s connected & bipartite.

**Lemma (Stationary Distribution of \( M_G \))**

For all \( v \in V \), \( \pi_v = \frac{d(v)}{2m} \).

**Proof.**

Let \( \Gamma(v) \) denote the graph neighborhood of \( v \):

\[
\sum_{u, v} \pi_u P_{u,v} = \sum_{u \in \Gamma(v)} \frac{d(u)}{2m} \cdot \frac{1}{d(u)} = \frac{d(v)}{2m} = \pi_v
\]

\( \square \)
Hitting Time

Definition
The hitting time $h_{uv}$ is the expected time taken by a random walk starting at node $u$ until it arrives at $v$.

Theorem
*For any edge $(u, v) \in E$, $h_{vu} < 2m$.*

Proof.
- By Fundamental Theorem: $h_{uu} = \pi_u^{-1} = 2m/d(u)$
- But we can also write $h_{uu}$ as

$$h_{uu} = \frac{1}{d(u)} \sum_{w \in \Gamma(u)} (1 + h_{wu})$$

- Hence,

$$2m = \sum_{w \in \Gamma(u)} (1 + h_{wu}) > h_{vu}$$
Cover Time (1/2)

Definition
$C_u(G)$ denote the expected time to visit every node after starting at $u$. The cover time of $G$ is $C(G) = \max_u C_u(G)$.

Theorem
$C(G) < 4mn$ where $|V| = n$ and $|E| = m$.

Corollary
Expected time of at 2-SAT algorithm is $O(n^2)$. 
Proof.

Let $T$ be a spanning tree of $G$ and $v_0 \in V$.

Let $v_0, v_1, \ldots, v_{2n-2} = v_0$ be the vertices visited in a traversal of $T$ that traverses each edge exactly once.

Then

$$C_{v_0}(G) \leq \sum_{j=0}^{2n-3} h_{v_j, v_{j+1}}$$

We know $h_{v_j, v_{j+1}} \leq 2m$ because $(u, v)$ is an edge.

Hence, $C_{v_0}(G) \leq 2(n - 1)2m$ for all $v_0$. 

\[\square\]
Readings

For next time, please make sure you’ve read:

- Chapter 4: 4.3 (4 pages)
- Chapter 5: 5.1, 5.2, 5.5 (12 pages)