Outline

Wiring Problem

Maximum Satisfiability

Puzzle
Global Wiring in Gate-Arrays

- $\sqrt{n} \times \sqrt{n}$ array of adjacent square gates.
- $t$ pairs of terminals need connected.
- Add a wire to connect each pair.
- **Constraint**: Wires only travel horizontally and/or vertically and may only “bend” once.
- **Problem**: Want to minimize the maximum number of wires that cross the border between any two adjacent gates.

Let’s see a picture...  
- How many borders?  
- How many possible wirings?
Define
\[ x_{i0} = \begin{cases} 
1 & \text{if wire } i \text{ is first routed horizontally} \\
0 & \text{otherwise} 
\end{cases} \]
\[ x_{i1} = \begin{cases} 
1 & \text{if wire } i \text{ is first routed vertically} \\
0 & \text{otherwise} 
\end{cases} \]
\[ T_{b0} = \{ i : \text{wire } i \text{ is routed through border } b \text{ if } x_{i0} = 1 \} \]
\[ T_{b1} = \{ i : \text{wire } i \text{ is routed through border } b \text{ if } x_{i1} = 1 \} \]

We want to minimize:
\[ \max_{b} \left( \sum_{i \in T_{b0}} x_{i0} + \sum_{i \in T_{b1}} x_{i1} \right) \]
Formulate as 0/1 Linear Program

minimize \( w \)

where \( x_{i0}, x_{i1} \in \{0, 1\} \) for all \( i \in [t] \)

subject to \( x_{i0} + x_{i1} = 1 \) for all \( i \in [t] \)

\[
\sum_{i \in T_{b0}} x_{i0} + \sum_{i \in T_{b1}} x_{i1} \leq w \text{ for all } b
\]

- Unfortunately 0/1 linear programming is NP hard.
- One option is to relax the program to get a linear program:

  replace \( x_{i0}, x_{i1} \in \{0, 1\} \) by \( 0 \leq x_{i0}, x_{i1} \leq 1 \)

- Can efficiently find a solution to the relaxed program but how can we use it to get a solution to the original problem?
A factor 2 approximation

Let $\hat{w}$ be optimum solution to the relaxed program and let $w_0$ be optimum solution to the original program:

$$\hat{w} \leq w_0$$

For $i \in [t]$, let $\hat{x}_{i0}, \hat{x}_{i1}$ be solution to relaxed program:

$$\max(\hat{x}_{i0}, \hat{x}_{i1}) \geq 1/2$$

Round the larger up to 1 and smaller down to 0.

Let $\tilde{x}_{i0}, \tilde{x}_{i1}$ be the new values.

Then, for all borders $b$:

$$\sum_{i \in T_{b0}} \tilde{x}_{i0} + \sum_{i \in T_{b1}} \tilde{x}_{i1} \leq 2 \left( \sum_{i \in T_{b0}} \hat{x}_{i0} + \sum_{i \in T_{b1}} \hat{x}_{i1} \right) \leq 2\hat{w} \leq 2w_0$$
Randomized Rounding (1/2)

- For $i \in [t]$, let $\hat{x}_i$, $\hat{x}_1$, $\hat{w}$ be solution to relaxed program.
- With prob. $\hat{x}_0$, let $\tilde{x}_i = 1$ & $\tilde{x}_i = 0$. Otherwise, set $\tilde{x}_0 = 0$ & $\tilde{x}_1 = 1$
- For any border $b$, let
  \[
  \tilde{w}(b) = \sum_{i \in T_{b_0}} \tilde{x}_i + \sum_{i \in T_{b_1}} \tilde{x}_i
  \]
- Then,
  \[
  \mathbb{E}[\tilde{w}(b)] = \sum_{i \in T_{b_0}} \hat{x}_i + \sum_{i \in T_{b_1}} \hat{x}_i \leq \hat{w}
  \]
Randomized Rounding (2/2)

- Note that $\tilde{w}(b)$ is the sum of independent Poisson trials.
- Using the Chernoff bound and $\mathbb{E}[\tilde{w}(b)] \leq w_0$:

$$\mathbb{P}[\tilde{w}(b) \geq (1 + \delta)w_0] \leq \left[ \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right]^{w_0}$$

- Apply the union bound:

$$\mathbb{P}[\exists b : \tilde{w}(b) \geq (1 + \delta)w_0] < 2n \left[ \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right]^{w_0}$$

- With probability at least $1 - \epsilon$

$$\max_b (\tilde{w}(b)) \leq \left( 1 + \sqrt{\frac{4 \ln(2n/\epsilon)}{w_0}} \right)^{w_0}$$

if $w_0 \geq \omega(\log(n/\epsilon))$
Maximum Satisfiability

- **Input:** \( m \) clauses in \( n \) Boolean variables \( x_1, \ldots, x_n \), e.g.,

\[
\begin{align*}
x_1 \lor x_2 \lor \bar{x}_3, & \quad \bar{x}_1 \lor \bar{x}_4, & \quad \ldots, & \quad x_9 \lor x_{10} \lor x_{21}
\end{align*}
\]

where \( \bar{x}_i = 1 - x_i \).

- **Goal:** Maximize the number of clauses that are satisfied.

**Notes:**

- Problem is NP-hard so will consider approximation algorithms.
- A randomized algorithm is an \( \alpha \)-approximations for a maximization problem if it returns \( X \) with \( \mathbb{E}[X]/\text{OPT} \geq \alpha \).
- Call \( x_i \) and \( \bar{x}_i \) literals. Assume \( x_i \) and \( \bar{x}_i \) don't appear in same clause.
Easy $1/2$-approximation

Theorem

A truth assignment chosen uniformly at random is a $1/2$ approx.

Proof.

- Independently set each $x_i$ to 0 with prob. $1/2$ and 1 otherwise.
- Let $Z_i = 1$ if $i$-th clause is satisfied and 0 otherwise.
- If $i$-th clause has $k$ literals,

$$\mathbb{P}[Z_i = 1] = 1 - 1/2^k \geq 1/2$$

- Expected number of clauses satisfied is $\mathbb{E}\left[\sum_{i \in [m]} Z_i\right] \geq m/2$

Example of the Probabilistic Method: Above prove shows that there always exists a way to satisfy $m/2$ clauses. This is tight.
Formulate as Linear Program

Define the following sets:

\[ C_j = \{ i : x_i \text{ or } \bar{x}_i \text{ appears in } j\text{-th clause} \} \]
\[ C_j^+ = \{ i : x_i \text{ appears in } j\text{-th clause} \} \]
\[ C_j^- = \{ i : \bar{x}_i \text{ appears in } j\text{-th clause} \} \]

Consider the following program:

\[
\text{maximize } \sum_{j=1}^{m} z_j \\
\text{where } x_i, z_j \in \{0, 1\} \text{ for all } i \in [n], j \in [m] \\
\text{subject to } \sum_{i \in C_j^+} x_i + \sum_{i \in C_j^-} (1 - x_i) \geq z_j \text{ for all } j \in [m]
\]

Relax by replacing "\(x_i, z_j \in \{0, 1\}\)" by "\(0 \leq x_i, z_j \leq 1\)"
Let $\hat{x}_i, \hat{z}_j$ be solutions to the relaxed program.

With probability $\hat{x}_i$, set $x_i = 1$ and $x_i = 0$ otherwise.

By independence,

$$\mathbb{P}[Z_j = 1] = 1 - \prod_{i \in C_j^+} (1 - \hat{x}_i) \cdot \prod_{i \in C_j^-} \hat{x}_i$$

Since $\sum_{i \in C_j^+} \hat{x}_i + \sum_{i \in C_j^-} (1 - \hat{x}_i) \geq \hat{z}_j$, if $C_j$ has $k$ literals

$$\mathbb{P}[Z_j = 1] \geq 1 - (1 - \hat{z}_j/k)^k$$

For $\beta_k = 1 - (1 - 1/k)^k$:

$$\mathbb{P}[Z_j = 1] \geq 1 - (1 - \hat{z}_j/k)^k \geq \beta_k \hat{z}_j \geq (1 - 1/e)\hat{z}_j$$

$$\mathbb{E} \left[ \sum_{j \in [m]} Z_j \right] \geq (1 - 1/e) \sum_{j \in [m]} \hat{z}_j \geq (1 - 1/e) \text{OPT}$$
Recap

- **First algorithm:**
  - Set each variable independently to 0 or 1 with equal probability.
  - Gave $1/2$ approximation.
  - Would have been better if all clauses had many literals.

- **Second algorithm:**
  - Set each variable independently to 0 or 1 with probability based on
    the linear program solution.
  - Gave $(1 - 1/e)$ approximation.
  - Would have been better if all clauses had few literals.

- **What would be a good third algorithm?**
3/4 Approximation

Theorem

Running both algorithms and returning best solution gives 3/4 approximation.

Proof.

Let $A$ and $B$ be number of clauses satisfied by first and second algorithms: $\mathbb{E} \left[ \max(A, B) \right] \geq \mathbb{E} [A] / 2 + \mathbb{E} [B] / 2$.

Let $S_k = \{ j : j\text{-th clause contains exactly } k\text{ literals} \}$

From the analysis of the previous algorithms:

$$A = \sum_k \sum_{j \in S_k} (1 - 2^{-k}) \quad \text{and} \quad B \geq \sum_k \sum_{j \in S_k} \beta_k \hat{z}_j$$

Therefore, using $1 - 2^{-k} + \beta_k \geq 3/2$ for all $k$:

$$\mathbb{E} [A] / 2 + \mathbb{E} [B] / 2 \geq \sum_k \sum_{j \in S_k} \frac{1 - 2^{-k} + \beta_k}{2} \hat{z}_j \geq 0.75 \text{OPT}$$
Outline

Wiring Problem

Maximum Satisfiability

Puzzle
Puzzle

- You lose the unbiased coin with which you were planning to do the homework.
- A friend lends you a biased coin but doesn’t tell you the bias.
- Without trying to estimate the bias, how can you use the biased coin to simulate a perfectly unbiased coin?