Recap: Counting Independent Sets

- If we can sample independent sets efficiently, then we can count the number, $|\Omega(G)|$, of independent sets efficiently as follows...
- Let $G_m = G$ and construct $G_{i-1}$ be removing an arbitrary edge from $G_i$. The $\Omega(G_i)$ be the set of independent sets of $G_i$.
- Then,
  \[
  |\Omega(G)| = \frac{|\Omega(G_m)|}{|\Omega(G_{m-1})|} \times \ldots \times \frac{|\Omega(G_1)|}{|\Omega(G_0)|} \times |\Omega(G_0)|
  \]
- Let $r_i = |\Omega(G_i)|/|\Omega(G_{i-1})|$ and note that $|\Omega(G_0)| = 2^n$:
  \[
  |\Omega(G)| = 2^n \prod_{i \in [m]} r_i
  \]
- Hence if we find estimates $\tilde{r}_i$ such that
  \[
  (1 - \epsilon/m)r_i \leq \tilde{r}_i \leq (1 + \epsilon/(2m))r_i
  \]
  then $2^n \prod_{i \in [m]} \tilde{r}_i$ is a $(1 + \epsilon)$ approximation of $|\Omega(G)|$
Recap: Estimating $r_i$

Note that

$$1 \geq r_i = \frac{|\Omega(G_i)|}{|\Omega(G_{i-1})|} = \frac{|\Omega(G_i)|}{|\Omega(G_i)| + |\Omega(G_{i-1}) \setminus \Omega(G_i)|} \geq 1/2$$

where last line follows since every set in $\Omega(G_{i-1}) \setminus \Omega(G_i)$ includes both end points of the edge removed to generate $G_{i-1}$ and so

$$|\Omega(G_{i-1}) \setminus \Omega(G_i)| \leq |\Omega(G_i)|$$

Hence, if we sample

$$(3 \ln(2/\delta))/((\epsilon/(2m))^2 r_i) \leq 24m^2 \ln(2/\delta))/\epsilon^2$$

independent sets uniformly from $\Omega(G_{i-1})$ and return the fraction that are also independent sets in $G_i$, we get a $1 + \epsilon/(2m)$ approximation for $r_i$ with probability $1 - \delta$.

Setting $\delta = 1/m^2$ ensures we get good enough estimates for all $r_1, r_2, \ldots, r_m$ with probability at least $1 - 1/m$. 

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But how do we sample independent sets?

- Design a Markov chain whose state space is the set of elements we want to sample and whose stationary distribution is the distribution we want to sample from.
- Let $X_0, X_1, X_2, \ldots$, be a run of the Markov chain. For sufficiently large $r$, $X_r, X_{2r}, X_{3r}, \ldots$ are nearly independent samples from the required distribution.
- . . . this is Markov Chain Monte Carlo Method.
Design a Markov Chain

Want to design Markov chain whose states $\Omega$ are exactly the independent sets and whose stationary distribution is uniform.

First attempt:
- Let the states be the independent sets of a graph.
- If the current state is independent set $X_t$:
  - Let $X_{t+1}$ be chosen uniformly at randomly from
    $$N(X_t) = \{Y : \text{independent set formed by adding/removing a node from } X_t\}$$

Problem is that stationary distribution might not be uniform!
Making Stationary Distribution Uniform

Lemma
Consider Markov Chain with finite state space \( \Omega \) and transition matrix \( P \). For each state \( i \) define \( N(i) = \{ j : P_{i,j} > 0 \} \) and \( M \geq \max |N(j)| \). Then, Markov chain \( (\Omega, Q) \) where

\[
Q_{i,j} = \begin{cases} 
1/M & \text{if } i \neq j \text{ and } j \in N(i) \\
0 & \text{if } i \neq j \text{ and } j \notin N(i) \\
1 - |N(i)|/M & \text{if } i = j
\end{cases}
\]

has an uniform stationary distribution.

Proof.
If \( \pi_1 = \pi_2 = \pi_3 \ldots \) then

\[
[\pi Q]_i = \pi_i (1 - |N(i)|/M) + \sum_{j \in N(i): j \neq i} \pi_j / M = \pi_i
\]
New Markov Chain for Independent Sets

New attempt:

- Let the states be the independent sets of a graph.
- If the current state is independent set $X_t$:
  - Pick random vertex $v$ from $V$
  - If $v \in X_t$, let $X_{t+1} = X_t \setminus \{v\}$.
  - If $v \notin X_t$ and $X_t \cup \{v\}$ is independent, let $X_{t+1} = X_t \cup \{v\}$.
  - Otherwise $X_{t+1} = X_t$.

Note that the transition probability between any different states is $1/|V|$ and previous lemma implies stationary distribution is uniform.
Metropolis Algorithm

Suppose we want a stationary distribution that isn’t uniform?

Theorem

For a finite state space $\Omega$ and neighborhoods $N(i)$ for each state $i$, let $M \geq \max_i N(i)$. Then a Markov chain with transition matrix

$$Q_{i,j} = \begin{cases} 
\min(1, \pi_j/\pi_i)/M & \text{if } i \neq j \text{ and } j \in N(i) \\
0 & \text{if } i \neq j \text{ and } j \notin N(i) \\
1 - \sum_{k \in N(i): k \neq i} \min(1, \pi_k/\pi_i)/M & \text{if } i = j
\end{cases}$$

has the stationary distribution where $\pi_i$ is the weight of state $i$.

Proof.

$$[\pi Q]_i = \pi_i \left(1 - \sum_{k \in N(i): k \neq i} \frac{\min(1, \pi_k/\pi_i)}{M}\right) + \sum_{j \in N(i): j \neq i} \pi_j \cdot \frac{\min(1, \pi_i/\pi_j)}{M}$$

$$= \pi_i + \sum_{j \in N(i): j \neq i} \left(\pi_j \cdot \frac{\min(1, \pi_i/\pi_j)}{M} - \pi_i \cdot \frac{\min(1, \pi_j/\pi_i)}{M}\right) = \pi_i$$
Weighting the Independent Sets

For independent set $X$, suppose we want

$$
\pi_X = \frac{\lambda^{|X|}}{\sum_{Y \text{independent}} \lambda^{|Y|}}
$$

- Let the states be the independent sets of a graph.
- If the current state is independent set $X_t$:
  - Pick random vertex $v$ from $V$.
  - If $v \in X_t$, let $X_{t+1} = X_t \setminus \{v\}$ with probability $\min(1, 1/\lambda)$.
  - If $v \notin X_t$ and $X_t \cup \{v\}$ is independent, let $X_{t+1} = X_t \cup \{v\}$ with probability $\min(1, \lambda)$.
  - Otherwise $X_{t+1} = X_t$.

Note that the transition probability between different states has the required format for the previous theorem to apply.