CMPSCI 611: Advanced Algorithms
Lecture 14: Min-Cut

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From Last Time: Quicksort

Problem: Sort an array of distinct values $X = [x_1, \ldots, x_n]$

Algorithm

1. Pick a pivot $x \in X$ at random from the array
2. Construct new arrays $Y = [y_1, \ldots, y_k]$, $Z = [z_1, \ldots, z_{n-k-1}]$ where $y < x < z$ for all $y \in Y, z \in Z$
3. Recursively sort $Y$ and $Z$ to get $Y'$ and $Z'$
4. Return the array that concatenates $Y'$, $x$, and $Z'$

What’s the expected number of comparisons performed in this algorithm?
Probability two items are compared

Lemma
Let \( a \) and \( b \) be the \( i \)-th and \( j \)-th smallest element of \( X \) where \( i < j \).

\[
\Pr[a \text{ is compared to } b] = \frac{2}{j - i + 1}
\]

Proof.
1. Consider \( S = \{ x \in X : a \leq x \leq b \} \)
2. \( a \) and \( b \) are compared iff the first pivot chosen from \( S \) is either \( a \) or \( b \)
3. Elements of \( S \) are equally likely to be chosen as a pivot, so

\[
\Pr[a \text{ is compared to } b] = \frac{2}{|S|} = \frac{2}{j - i + 1}
\]
Expected Number of Comparisons

Lemma

Expected number of comparisons performs is \( O(n \log n) \).

Proof.

1. Let \( Z_{ij} = 1 \) if the \( i \)-th smallest element is compared to \( j \)-th smallest element and \( Z_{ij} = 0 \) otherwise.
2. Number of comparisons: \( \sum_{1 \leq i < j \leq n} Z_{ij} \)
3. Expected number of comparisons:

\[
\mathbb{E} \left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] = \sum_{1 \leq i < j \leq n} \mathbb{E} [Z_{ij}] = \sum_{1 \leq i < j \leq n} \frac{2}{j - i + 1} = \sum_{j=2}^{n} \sum_{k=2}^{j} \frac{2}{k}
\]

4. Because \( H_n = 1 + 1/2 + 1/3 + \ldots + 1/n = O(\log n) \),

\[
\mathbb{E} \left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] \leq \sum_{j=2}^{n} \sum_{k=2}^{j} \frac{2}{k} = n \cdot O(\log n) = O(n \log n)
\]
Outline

Karger’s Randomized Min-Cut Algorithm
Min-Cut Problem

Given an unweighted, multi-graph $G = (V, E)$, we want to partition $V$ into $V_1$ and $V_2$ such that $|E \cap (V_1 \times V_2)|$ is minimized.

Algorithm

- **Contract** a random edge $e = (u, v)$ and remove self-loops but not multi-edges
- Repeat until there are only 2 vertices remaining.
- Output the number of remaining edges.

Let $|V| = n$ and $|E| = m$. 
Example
Correctness with low probability

**Theorem**

*Algorithm is correct with probability* \( \geq \frac{2}{n^2} \) *and never underestimates.*

**Proof.**

- Min cut of the graph doesn’t decrease: after \( e = (x, y) \) contracted, set of possible cuts is limited to all those with \( x \) and \( y \) on same side
- Let \( C = (V_1, V_2) \) be a specific minimum cut with \( |C| = k \).
- Let \( A_i \) be event that we don’t contract edge across \( C \) at step \( i \).

\[
\mathbb{P} \left[ \cap_{1 \leq i \leq n-2} A_i \right] = \mathbb{P} [A_1] \mathbb{P} [A_2 | A_1] \ldots \mathbb{P} [A_{n-2} | \cap_{1 \leq i \leq n-3} A_i] 
\]

- Number of edges before \( i \)-th step if no edges across \( C \) have been contracted so far is at least \( (n - i + 1)k/2 \) since there are \( n - i + 1 \) nodes remaining each with degree \( \geq k \)
- \( \mathbb{P} [A_i | A_1 \cap A_2 \cap \ldots \cap A_{i-1}] \geq 1 - \frac{2}{(n - i + 1)} \) and so

\[
\mathbb{P} \left[ \cap_{1 \leq i \leq n-2} A_i \right] \geq (1 - \frac{2}{n})(1 - \frac{2}{n-1})(1 - \frac{2}{n-2}) \ldots (1 - \frac{2}{3}) = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \ldots \cdot \frac{1}{3} = \frac{2}{n(n-1)}
\]
Min-Cut Problem: Boosting the probability

Theorem
Repeating $\alpha n^2/2$ times (with new random coin flips) and returning smallest cut is correct with probability at least $1 - e^{-\alpha}$.

Proof.
- Because each repeat is independent,

$$P[\text{always fails}] = \prod_{1 \leq i \leq \alpha n^2/2} P[i\text{-th try fails}] \leq (1 - 2/n^2)^{\alpha n^2/2}$$

- Use fact $1 - x \leq e^{-x}$ for $x \geq 0$ and simplify.