Problem: Sort an array of distinct values \( X = [x_1, \ldots, x_n] \)
From Last Time: Quicksort

Problem: Sort an array of distinct values $X = [x_1, \ldots, x_n]$

Algorithm

1. Pick a pivot $x \in X$ at random from the array
2. Construct new arrays $Y = [y_1, \ldots, y_k]$, $Z = [z_1, \ldots, z_{n-k-1}]$ where $y < x < z$ for all $y \in Y, z \in Z$
3. Recursively sort $Y$ and $Z$ to get $Y'$ and $Z'$
4. Return the array that concatenates $Y'$, $x$, and $Z'$

What's the expected number of comparisons performed in this algorithm? $\frac{2}{9}$
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What’s the expected number of comparisons performed in this algorithm?
Probability two items are compared

**Lemma**
Let \(a\) and \(b\) be the \(i\)-th and \(j\)-th smallest element of \(X\) where \(i < j\).

\[
\Pr[a \text{ is compared to } b] = \frac{2}{j - i + 1}
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Proof.
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Proof.
1. Consider $S = \{x \in X : a \leq x \leq b\}$
Probability two items are compared

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Proof.
1. Consider $S = \{x \in X : a \leq x \leq b\}$
2. $a$ and $b$ are compared iff the first pivot chosen from $S$ is either $a$ or $b$
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1. Consider $S = \{x \in X : a \leq x \leq b\}$
2. $a$ and $b$ are compared iff the first pivot chosen from $S$ is either $a$ or $b$
3. Elements of $S$ are equally likely to be chosen as a pivot, so

$$\Pr[a \text{ is compared to } b] = \frac{2}{|S|}$$
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Expected Number of Comparisons

Lemma

*Expected number of comparisons performs is* $O(n \log n)$. 

Proof.

1. Let $Z_{ij} = 1$ if the $i$-th smallest element is compared to $j$-th smallest element and $Z_{ij} = 0$ otherwise.

2. Number of comparisons:

\[
\sum_{1 \leq i < j \leq n} Z_{ij}
\]

3. Expected number of comparisons:

\[
E\left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right]
\]

\[
= \sum_{1 \leq i < j \leq n} E[Z_{ij}]
\]

\[
= \sum_{1 \leq i < j \leq n} 2^{j-i} + 1
\]

\[
= \sum_{j=2}^{n} \sum_{k=2}^{j} 2^{k-1}
\]

\[
= n \cdot O(\log n)
\]

Because

\[
H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = O(\log n),
\]

\[
E\left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] \leq n \sum_{j=2}^{n} \sum_{k=2}^{j} 2^{k-1}
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$$
\mathbb{E} \left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] = \sum_{1 \leq i < j \leq n} \mathbb{E} [Z_{ij}]
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2. Number of comparisons: \(\sum_{1 \leq i < j \leq n} Z_{ij}\)
3. Expected number of comparisons:

\[
\mathbb{E} \left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] = \sum_{1 \leq i < j \leq n} \mathbb{E}[Z_{ij}] = \sum_{1 \leq i < j \leq n} \frac{2}{j - i + 1} = \sum_{j=2}^{n} \sum_{k=2}^{j} \frac{2}{k}
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Expected Number of Comparisons

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$$
E \left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] = \sum_{1 \leq i < j \leq n} E[Z_{ij}] = \sum_{1 \leq i < j \leq n} \frac{2}{j - i + 1} = \sum_{j=2}^{n} \sum_{k=2}^{j} \frac{2}{k}
$$

4. Because $H_n = 1 + 1/2 + 1/3 + \ldots + 1/n = O(\log n)$,

$$
E \left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] \leq \sum_{j=2}^{n} \sum_{k=2}^{n} \frac{2}{k} = n \cdot O(\log n) = O(n \log n)
$$
Outline

Karger’s Randomized Min-Cut Algorithm
Min-Cut Problem

Given an unweighted, multi-graph $G = (V, E)$, we want to partition $V$ into $V_1$ and $V_2$ such that $|E \cap (V_1 \times V_2)|$ is minimized.
Min-Cut Problem

Given an unweighted, multi-graph \( G = (V, E) \), we want to partition \( V \) into \( V_1 \) and \( V_2 \) such that \( |E \cap (V_1 \times V_2)| \) is minimized.

Algorithm

- **Contract** a random edge \( e = (u, v) \) and remove self-loops but not multi-edges
- **Repeat** until there are only 2 vertices remaining.
- **Output** the number of remaining edges.
Min-Cut Problem

Given an unweighted, multi-graph $G = (V, E)$, we want to partition $V$ into $V_1$ and $V_2$ such that $|E \cap (V_1 \times V_2)|$ is minimized.

Algorithm

- **Contract** a random edge $e = (u, v)$ and remove self-loops but not multi-edges
- Repeat until there are only 2 vertices remaining.
- Output the number of remaining edges.

Let $|V| = n$ and $|E| = m$. 
Example

Step 1

Step 2

Step 3

Step 4

Step 5

Step 6
Correctness with low probability

Theorem

*Algorithm is correct with probability \( \geq \frac{2}{n^2} \) and never underestimates.*
Correctness with low probability

Theorem
Al\textit{gorithm is correct with probability }\geq 2/n^2 \textit{ and never underestimates.}

Proof.

\begin{itemize}
\item Min cut of the graph doesn't decrease: after \( e = (x, y) \) contracted, set of possible cuts is limited to all those with \( x \) and \( y \) on same side
\end{itemize}
Correctness with low probability

Theorem

Algorithm is correct with probability $\geq 2/n^2$ and never underestimates.

Proof.

- Min cut of the graph doesn’t decrease: after $e = (x, y)$ contracted, set of possible cuts is limited to all those with $x$ and $y$ on same side
- Let $C = (V_1, V_2)$ be a specific minimum cut with $|C| = k$. 
Correctness with low probability

**Theorem**

*Algorithm is correct with probability $\geq \frac{2}{n^2}$ and never underestimates.*

**Proof.**

- Min cut of the graph doesn’t decrease: after $e = (x, y)$ contracted, set of possible cuts is limited to all those with $x$ and $y$ on same side
- Let $C = (V_1, V_2)$ be a specific minimum cut with $|C| = k$.
- Let $A_i$ be event that we don’t contract edge across $C$ at step $i$.

\[
P[\cap_{1 \leq i \leq n-2} A_i] = P[A_1] P[A_2|A_1] \cdots P[A_{n-2}| \cap_{1 \leq i \leq n-3} A_i]
\]
Correctness with low probability

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$$\Pr[\cap_{1 \leq i \leq n-2} A_i] = \Pr[A_1] \Pr[A_2 | A_1] \ldots \Pr[A_{n-2} | \cap_{1 \leq i \leq n-3} A_i]$$

- Number of edges before $i$-th step if no edges across $C$ have been contracted so far is at least
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**Theorem**
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**Proof.**
- Min cut of the graph doesn’t decrease: after $e = (x, y)$ contracted, set of possible cuts is limited to all those with $x$ and $y$ on same side.
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\]

- Number of edges before $i$-th step if no edges across $C$ have been contracted so far is at least $(n - i + 1)k/2$ since there are $n - i + 1$ nodes remaining each with degree $\geq k$.
Correctness with low probability

**Theorem**

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- $P[A_i|A_1 \cap A_2 \cap \ldots \cap A_{i-1}] \geq 1 - 2/(n - i + 1)$
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$$P[\bigcap_{1 \leq i \leq n-2} A_i] = P[A_1] P[A_2|A_1] \cdots P[A_{n-2}|\bigcap_{1 \leq i \leq n-3} A_i]$$

- Number of edges before $i$-th step if no edges across $C$ have been contracted so far is at least $(n - i + 1)k/2$ since there are $n - i + 1$ nodes remaining each with degree $\geq k$.
- $P[A_i|A_1 \cap A_2 \cap \ldots \cap A_{i-1}] \geq 1 - 2/(n - i + 1)$ and so

$$P[\bigcap_{1 \leq i \leq n-2} A_i] \geq \left(1 - \frac{2}{n}\right) \left(1 - \frac{2}{n-1}\right) \left(1 - \frac{2}{n-2}\right) \cdots \left(1 - \frac{2}{3}\right)$$
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\mathbb{P}[\bigcap_{1 \leq i \leq n-2} A_i] = \mathbb{P}[A_1] \mathbb{P}[A_2|A_1] \ldots \mathbb{P}[A_{n-2} | \bigcap_{1 \leq i \leq n-3} A_i]
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$$
\mathbb{P}[\bigcap_{1 \leq i \leq n-2} A_i] \geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right) \ldots \left(1 - \frac{2}{3}\right)
$$

$$
= \frac{n-2}{n} \cdot \frac{n-3}{8n-1} \cdot \frac{n-4}{n-2} \cdot \ldots \cdot \frac{1}{3} = \frac{2}{n(n-1)}
$$
Min-Cut Problem: Boosting the probability

Theorem
Repeating $\alpha n^2/2$ times (with new random coin flips) and returning smallest cut is correct with probability at least $1 - e^{-\alpha}$. 

Proof. Because each repeat is independent, $P[\text{always fails}] = \prod_{1 \leq i \leq \alpha} P[i\text{-th try fails}] 
\leq (1 - 2/n^2)^{\alpha n^2/2}$ 

Use fact $1 - x \leq e^{-x}$ for $x \geq 0$ and simplify.
Min-Cut Problem: Boosting the probability

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Proof.

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  \[ P [\text{always fails}] = \prod_{1 \leq i \leq \alpha n^2 / 2} P [i\text{-th try fails}] \leq (1 - 2/n^2)^{\alpha n^2 / 2} \]

- Use fact $1 - x \leq e^{-x}$ for $x \geq 0$ and simplify.

\[ \square \]