Residual
Augmenting Path
New Residual Graph
Ford-Fulkerson Algorithm with Edmonds-Karp Heuristic

Algorithm

1. flow $f = 0$
2. while there exists an augmenting path $p$ for $f$
   
   2.1 find shortest (unweighted) augmenting path $p$
   
   2.2 augment $f$ by $b(p)$ units along $p$

3. return $f$

Theorem

The algorithms finds a maximum flow in time $O(|E|^2|V|)$
Proof of Running Time (1/3)

Definition
Let $\delta_f(s, u)$ be length of shortest unweighted path from $s$ to $u$ in the $G_f$.

Definition
$(u, v)$ is critical if it’s on augmenting path $p$ for $f$ and $C_f(u, v) = b(p)$.

Lemma
$\delta_f(s, v)$ is non-decreasing as $f$ changes.

Lemma
Between occasions when $(u, v)$ is critical, $\delta_f(s, u)$ increases by at least 2.

Proof of Running Time.

- Max distance in $G_f$ is $|V|$ so any edge is critical at most $1 + |V|/2$ times
- At most $2|E|$ edges in residual network
- There’s a critical edge in each iteration so $O(|E||V|)$ iterations
- Each iteration takes $O(|E|)$ to find shortest path

□
Proof of Running Time (2/3)

Lemma
\( \delta_f(s, v) \) is non-decreasing as \( f \) changes.

Proof.

▶ Consider augmenting \( f \) to \( f' \)
▶ For contradiction, pick \( v \) that minimizes \( \delta_{f'}(s, v) \) subject to:

\[
\delta_{f'}(s, v) < \delta_f(s, v)
\]
▶ Let \( u \) be vertex before \( v \) on shortest \( s-v \) path in \( G_{f'} \). Note \( \delta_{f'}(s, u) \geq \delta_f(s, u) \) and \( \delta_{f'}(s, v) = \delta_{f'}(s, u) + 1 \)
▶ Claim \( (u, v) \notin E_f \)
   ▶ Otherwise \( \delta_f(s, v) \leq \delta_f(s, u) + 1 \)
   ▶ \( \delta_f(s, u) \leq \delta_{f'}(s, u) \) implies \( \delta_f(s, v) \leq \delta_{f'}(s, u) + 1 = \delta_{f'}(s, v) \).
   ▶ Contradicts \( \delta_{f'}(s, v) < \delta_f(s, v) \).
▶ \( (u, v) \notin E_f \) and \( (u, v) \in E_{f'} \) implies augmentation contains \( (v, u) \)
▶ Since augmentation was shortest path:

\[
\delta_f(s, v) = \delta_f(s, u) - 1 \leq \delta_{f'}(s, u) - 1 = \delta_{f'}(s, v) - 2
\]
Lemma

Between occasions when \((u, v)\) is critical, \(\delta_f(s, u)\) increases by at least 2.

Proof.

- Let \((u, v)\) be critical in the augmentation of \(f\)
- Since \((u, v)\) on shortest path: \(\delta_f(s, u) = \delta_f(s, v) - 1\)
- After augmentation \((u, v)\) disappears from residual network!
- Let \(f''\) be next flow with \((u, v) \in G_{f''}\) and \(f'\) be flow right before \(f''\)
- \((u, v) \notin G_{f'}\) but \((u, v) \in G_{f''}\) implies \((v, u)\) used to augment \(f'\)
- Therefore \(\delta_{f'}(s, v) = \delta_{f'}(s, u) - 1\) and so

\[
\delta_f(s, u) = \delta_f(s, v) - 1 \leq \delta_{f'}(s, v) - 1 = \delta_{f'}(s, u) - 2
\]
Probability Refresher

- **Expectation of random variable:**

\[
E[X] = \sum_r r P[X = r]
\]

- **Linearity of expectation:**

\[
E[X + Y] = E[X] + E[Y]
\]

- **Conditional Probability:** For arbitrary events \(A\) and \(B\),

\[
P[A|B] = \frac{P[A \cap B]}{P[B]}
\]

and

\[
P[\cap_{i=1}^n A_i] = P[A_1] P[A_2|A_1] \ldots P[A_n|\cap_{i=1}^{n-1} A_i]
\]
Quicksort

Problem: Sort an array of distinct values $X = [x_1, \ldots, x_n]$

Algorithm

1. *Pick a pivot* $x \in X$ *at random from the array*
2. *Construct new arrays* $Y = [y_1, \ldots, y_k]$, $Z = [z_1, \ldots, z_{n-k-1}]$ *where*
   \[ y < x < z \text{ for all } y \in Y, z \in Z \]
3. *Recursively sort* $Y$ and $Z$ *to get* $Y'$ and $Z'$
4. *Return the array that concatenates* $Y'$, $x$, and $Z'$

What’s the expected number of comparisons performed in this algorithm?
Probability two items are compared

Lemma
Let $a$ and $b$ be the $i$-th and $j$-th smallest element of $X$ where $i < j$.

$$\Pr[a \text{ is compared to } b] = \frac{2}{j - i + 1}$$

Proof.
1. Consider $S = \{x \in X : a \leq x \leq b\}$
2. $a$ and $b$ are compared iff the first pivot chosen from $S$ is either $a$ or $b$
3. Elements of $S$ are equally likely to be chosen as a pivot, so

$$\Pr[a \text{ is compared to } b] = \frac{2}{|S|} = \frac{2}{j - i + 1}$$
Expected Number of Comparisons

Lemma

*Expected number of comparisons performs is* $O(n \log n)$.

Proof.

1. Let $Z_{ij} = 1$ if the $i$-th smallest element is compared to $j$-th smallest element and $Z_{ij} = 0$ otherwise.

2. Number of comparisons: $\sum_{1 \leq i < j \leq n} Z_{ij}$

3. Expected number of comparisons:

$$
\mathbb{E} \left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] = \sum_{1 \leq i < j \leq n} \mathbb{E} [Z_{ij}] = \sum_{1 \leq i < j \leq n} \frac{2}{j - i + 1} = \sum_{j=2}^{n} \sum_{k=2}^{j} \frac{2}{k}
$$

4. Because $H_n = 1 + 1/2 + 1/3 + \ldots + 1/n = O(\log n)$,

$$
\mathbb{E} \left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] \leq \sum_{j=2}^{n} \sum_{k=2}^{n} \frac{2}{k} = n \cdot O(\log n) = O(n \log n)
$$

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