CMPSCI 611: Advanced Algorithms
Lecture 10: Seidel’s Algorithm

Andrew McGregor

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Seidel’s Algorithm

Problem: For an undirected, unweighted graph $G$, compute all distances.

Seidel’s Algorithm is based on matrix multiplication and runs in time

$$O(\mu(n) \log n)$$

where $\mu(n)$ is the time to multiply two $n \times n$ matrices together. Recall

$$n^2 \leq \mu(n) \leq n^{2.3727}$$

Definition

Let $M_G$ be the adjacency matrix of $G = (V, E)$, i.e., an $n \times n$ binary matrix where $M_G(i, j) = 1$ iff $(i, j) \in E$. 
The $G_2$ graph

Definition
Given a undirected, unweighted graph $G = (V, E)$, define $G_2 = (V, E')$ where $(i, j) \in E'$ iff $\delta_G(i, j) \leq 2$.

Lemma
Let $P_G(i, j) = 1$ if $\delta_G(i, j)$ is odd and $P_G(i, j) = 0$ otherwise. Then,

$$\delta_G(i, j) = 2\delta_{G_2}(i, j) - P_G(i, j).$$

Proof.
A path of length $2k$ in $G$ corresponds to a path of length $k$ in $G_2$. A path of length in $2k + 1$ in $G$ corresponds to a path of length $k + 1$ in $G_2$. □
Seidel’s Algorithm

Algorithm (Seidel($M_G$))

1. compute $M_{G_2}$
2. if all non-diagonal entries of $M_{G_2}(i,j)$ are 1, return $D_G$ where

$$D_G[i,j] = \begin{cases} 
0 & \text{if } i = j \\
1 & \text{if } M_G(i,j) = 1 \\
2 & \text{otherwise}
\end{cases}$$

3. else:
   3.1 compute $D_{G_2} = \text{Seidel}(M_{G_2})$
   3.2 compute $P_G$
   3.3 return $D_G = 2D_{G_2} - P_G$

Mystery Steps: How can we compute $M_{G_2}$ and $P_G$ efficiently?
The diameter of a graph $G$ is the "longest shortest path",

$$\text{diam}(G) = \max_{i,j} \delta_G(i,j)$$

Note that if $\text{diam}(G) \geq 3$:

$$\text{diam}(G_2) \leq \frac{\text{diam}(G)}{2} + \frac{1}{2} \leq \frac{2\text{diam}(G)}{3}$$

After recursing $t$ steps, the diameter is at most

$$(2/3)^t \text{diam}(G)$$

and so after $\log(n/2)/\log(3/2)$ steps, the diameter is at most 2.
Computing $M_{G^2}$ via $M_G \times M_G$

Lemma

$$M_{G^2}(i,j) = \begin{cases} 1 & \text{if } i \neq j \text{ and } (M_G(i,j) = 1 \text{ or } M^2_G(i,j) > 0) \\ 0 & \text{otherwise} \end{cases}$$

Proof.

$M^2_G(i,j) = \sum_k M_G(i,k)M_G(k,j) = \# \text{ of length 2 paths from } i \text{ to } j$. So there is an edge $(i,j)$ in $G_2$ iff $(i,j) \in G$ or $M^2_G(i,j) > 0$. \hfill \Box$

Can compute $M_{G^2}$ in $O(\mu(n))$ time.
Computing $P_G$ via $D_{G_2} \times M_G$

$P_G$ can be computed in $O(\mu(n))$ time...

Lemma

Let $X = D_{G_2}M_G$ where $D_{G_2}(i,j) = \delta_{G_2}(i,j)$. Then,

$$P_G(i,j) = 0 \iff \frac{X(i,j)}{\text{degree}_G(j)} \geq \delta_{G_2}(i,j)$$

where $\text{degree}_G(j)$ is the number of edges incident to node $j$ in graph $G$.

Note that,

$$\frac{X(i,j)}{\text{degree}_G(j)} = \frac{\sum_k \delta_{G_2}(i,k)M_G(k,j)}{\text{degree}_G(j)} = \frac{\sum_{k:\text{neighbor of } j \text{ in } G} \delta_{G_2}(i,k)}{\text{degree}_G(j)}$$

Fix $i$ and let $d_k = \delta_{G_2}(i,k)$, then we need to show:

$$P_G(i,j) = 0 \iff \text{ (average of } d_k \text{ over neighbors } k \text{ of } j \text{) } \geq d_j$$
Proof of Lemma

If $P_G(i, j) = 0$, then $\delta_G(i, j) = 2d_j$

For all neighbors $k$ note that $\delta_G(i, k)$ is either $2d_j - 1$, $2d_j$, or $2d_j + 1$

Hence, each $d_k$ is either $d_j$ or $d_j + 1$

Therefore average $d_k$ values is at least $d_j$

If $P_G(i, j) = 1$, then $\delta_G(i, j) = 2d_j - 1$

For all neighbors $k$ note that $\delta_G(i, k)$ is either $2d_j - 2$, $2d_j - 1$, or $2d_j$

Hence, each $d_k$ is either $d_j - 1$ or $d_j$

At least one neighbor has $\delta_G(i, k) = 2d_j - 2$ and $d_k = d_j - 1$

Therefore average $d_k$ values is strictly less than $d_j$
Total Running Time

Algorithm (Seidel($M_G$))

1. compute $M_G_2$
2. if $\forall i \neq j : M_G_2(i, j) = 1$, return

$$D_G(i, j) = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } M_G(i, j) = 1 \\
2 & \text{if otherwise}
\end{cases}$$

3. else:
   3.1 compute $D_G_2 = \text{Seidel}(M_G_2)$
   3.2 compute $P_G$
   3.3 return $D_G = 2D_G_2 - P_G$

Running Time: $O(\mu(n) \log n)$ since depth of recursion is $O(\log n)$ and each iteration takes $O(\mu(n))$ time.