Shortest Paths

Let $G = (V, E)$ be a directed graph with weights $w : E \to \mathbb{R}^+$. 

Definition
For path $p = (v_1, \ldots, v_k)$ be a path, define

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

The shortest path between $u$ and $v$ is

$$\delta(u, v) = \min\{w(p) : p \text{ is a path from $u$ to $v$}\}$$

if there is a path from $u$ to $v$ and $\infty$ otherwise.
Dijkstra’s Warm-Up

Single-Source Problem: Given $s \in V$, find $\delta(s, v)$ for all $v \in V$.

Dijkstra’s algorithm solves problem if all edges are non-negative:

- Maintains array $(d[v] : v \in V)$ where $d[v]$ will always be $\infty$ or the length of some path from $s$ to $v$, not necessarily the shortest. Hence,

  $$d[v] \geq \delta(s, v)$$

- Maintains a set of processed vertices $R$. We’ll prove that for all $v \in R$:

  $$d[v] = \delta(s, v)$$
Dijkstra's Algorithm

Algorithm

1. \( d[s] = 0 \) and for \( s \neq v \):
   \[
d[v] = w(s, v) \text{ if } (s, v) \in E \text{ and } \infty \text{ otherwise}
   \]

2. \( R \leftarrow \{s\} \)

3. While \( |R| < |V| \):
   3.1 \( u \leftarrow \arg\min_{v \notin R} d[v] \)
   3.2 \( R \leftarrow R + u \)
   3.3 For each \( v \notin R \) that is a neighbor of \( u \):
      \[
d[v] = \min(d[u] + w(u, v), d[v])
      \]

Running Time: \( O(|V|^2) \) for simple implementation but can be improved.
Example

1. Step 1: $d[s] = 0, d[a] = 3, d[b] = 6, d[c] = \infty$, and $R = \{s\}$

2. Step 2: $d[s] = 0, d[a] = 3, d[b] = 5, d[c] = 12$, and $R = \{s, a\}$

3. Step 3: $d[s] = 0, d[a] = 3, d[b] = 5, d[c] = 8$, and $R = \{s, a, b\}$

4. Step 4: $d[s] = 0, d[a] = 3, d[b] = 5, d[c] = 8$, and $R = \{s, a, b, c\}$
Correctness of Algorithm

The correctness of the algorithm follows because a) $d[v]$ never increases, b) $d[v] \geq \delta(s, u)$ at all times, and c) appealing to the following lemma:

**Lemma**

*When $u$ is added to $R$, $d[u] = \delta(s, u)$*
When \( u \) gets added to \( R \), \( d[u] \) is correct (1/2)

Let \( d_u[v] \) be value of \( d[v] \) just before \( u \) is chosen as minimum.

**Lemma**

*For all* \( u \), \( d_u[u] = \delta(s, u) \)

- **By contradiction**: Let \( u \) be first vertex put in \( R \) with \( d_u[u] > \delta(s, u) \)
- Consider a shortest path from \( s \) to \( u \). Let \( y \) be first vertex not in \( R \). Note that \( y \) may or may not be \( u \).

  - **Claim**: \( d_u[y] = \delta(s, y) \)
    - Let \( x \) be the predecessor of \( y \) on the path. Note that \( x \in R \).
    - \( d_x[x] = \delta(s, x) \) by assumption that \( u \) is first bad vertex.
    - After iteration where \( x \) is added to \( R \): \( d[y] \leq \delta(s, x) + w(x, y) \)
    - \( \delta(s, x) + w(x, y) = \delta(s, y) \) since path included shortest path to \( y \)
When $u$ gets added to $R$, $d[u]$ is correct (2/2)
Let $d_u[v]$ be value of $d[v]$ just before $u$ is chosen as minimum.

Lemma
For all $u$, $d_u[u] = \delta(s, u)$

- By contradiction: Let $u$ be first vertex put in $R$ with $d_u[u] > \delta(s, u)$
- Consider a shortest path from $s$ to $u$. Let $y$ be first vertex not in $R$. Note that $y$ may or may not be $u$.

- Claim: $d_u[y] = \delta(s, y)$
- Since $y$ lies on shortest path to $u$: $\delta(s, y) \leq \delta(s, u)$
- Putting above two lines together:
  $$d_u[y] = \delta(s, y) \leq \delta(s, u) < d_u[u]$$

- If $y \neq u$: Contradiction because $u$ was the next minimum and so
  $$d_u[u] \leq d_u[y]$$

- If $y = u$: Contradicts $d_u[y] < d_u[u]$