Outline

Divide and Conquer Template

Matrix Multiplication

Closest Pair of Points in a Plane
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Closest Pair of Points in a Plane
Divide and Conquer Methodology

- **General Goal:** Solve problem $P$ on an instance $I$ of “size” $n$.
- **Divide & Conquer:**
  1. Transform $I$ into smaller instances $I_1, \ldots, I_a$ each of “size” $n/b$
  2. Solve problem $P$ on each of $I_1, \ldots, I_a$ by recursion
  3. Combine the solutions to get a solution of $I$
- **Example (Merge Sort):** To sort $n$ numbers, divide into 2 sets of size $\frac{n}{2}$, sort each set, and merge.
Analyzing Divide and Conquer Algorithms

Let $T(n)$ be running time of algorithm on instance of size $n$. Then

$$T(1) = \Theta(1), \quad T(n) \leq aT(n/b) + O(n^\alpha)$$

where $O(n^\alpha)$ is time to create the subproblems and combine solutions.

Theorem (Master Theorem)

*If $a, b, \alpha$ are constants,*

$$T(n) = \begin{cases} 
O(n^\alpha) & \text{if } b^\alpha > a \\
O(n^{\log_b a}) & \text{if } b^\alpha < a \\
O(n^\alpha \log n) & \text{if } b^\alpha = a 
\end{cases}$$

Example (Merge Sort): $a = b = 2$ and $\alpha = 1$. Therefore the running time is $O(n \log n)$. 
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Matrix Multiplication

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First Attempt at Matrix Multiplication

Given two $n \times n$ matrices $A$ and $B$, multiply them together to get $C$:

$$c_{ij} = \sum_{k \in [n]} a_{ik} b_{kj}$$

Naive algorithm works in $O(n^3)$ time. Try Divide and Conquer...

- Divide $A$ and $B$ into four $n/2 \times n/2$ sub-matrices:

  $$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

- And note

  $$C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} = \begin{pmatrix} P_1 + P_2 & P_3 + P_4 \\ P_5 + P_6 & P_7 + P_8 \end{pmatrix}$$

  where $P_1 = A_{11}B_{11}$ and $P_2 = A_{12}B_{21}$ etc.

- Bad News: $T(n) = 8T(n/2) + \Theta(n^2)$ gives $T(n) = \Theta(n^3)$
Along comes Volker Strassen in 1969...
Strassen’s Algorithm

Break the problem into 7 sub-problems:

\[
\begin{align*}
P_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
P_2 &= (A_{21} + A_{22})(B_{11}) \\
P_3 &= (A_{11})(B_{12} - B_{22}) \\
P_4 &= (A_{22})(-B_{11} + B_{21}) \\
P_5 &= (A_{11} + A_{12})(B_{22}) \\
P_6 &= (-A_{11} + A_{21})(B_{11} + B_{12}) \\
P_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{align*}
\]

Then

\[
AB = \begin{pmatrix}
P_1 + P_4 - P_5 + P_7 & P_3 + P_6 \\
P_2 + P_4 & P_1 - P_2 + P_3 + P_6
\end{pmatrix}
\]

Good: \( T(n) = 7T\left(\frac{n}{2}\right) + \Theta(n^2) \) gives \( T(n) = \Theta(n^{2.81}) \).

Improvements: \( O(n^{2.376}) \) by Coppersmith, Winograd 1990, \( O(n^{2.3736}) \) by Stothers 2010, \( O(n^{2.3729}) \) by Williams 2011, \( O(n^{2.3728}) \) by Le Gall 2014, \( \ldots O(n^{2.371552}) \) by Williams, Xu, Xu, and Zhou in 2024.
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Closest Pair of Points in a Plane
Finding Minimum Distance between Points on a Plane

Problem: Given $n$ distinct points $p_1, \ldots, p_n \in \mathbb{R}^2$, find

$$\text{minimum distance between any two points} = \min_{i \neq j} d(p_i, p_j)$$

How long does naive algorithm take? $O(n^2)$

We’ll do it in $O(n \log n)$ steps.

For simplicity, assume no two points have the same $x$ or $y$ coordinate.
Minimum Distance Algorithm

1. Divide points $P$ with a vertical line into $P_L$ and $P_R$ where

   $$|P_L| = |P_R| = n/2$$

2. Recursively find minimum distance within $P_L$ and $P_R$:

   \[
   \delta_L = \min_{p,q \in P_L: p \neq q} d(p, q)
   \]

   \[
   \delta_R = \min_{p,q \in P_R: p \neq q} d(p, q)
   \]

3. Compute $\delta_M = \min_{p \in P_L, q \in P_R} d(p, q)$ and return

   $$\min(\delta_L, \delta_R, \delta_M)$$

Note: If Step 3 takes $O(n^2)$ time, we get

$$T(n) \leq 2T(n/2) + O(n^2) \implies T(n) = O(n^2)$$

If we can do Step 3 in $\Theta(n)$ time, we get $T(n) = O(n \log n)$. 
Making Step 3 Efficient

- Need to find \( \min(\delta_L, \delta_R, \delta_M) \) where \( \delta_M = \min_{p \in P_L, q \in P_R} d(p, q) \)
- Suppose that the dividing line is \( x = m \) and \( \delta = \min(\delta_L, \delta_R) \)
- Once we know \( \delta \), only need \( O(n) \) comparisons to find \( \min(\delta, \delta_M) \)
  1. Only compare \( p = (p_1, p_2) \) to \( q = (q_1, q_2) \) if \( p_2 \leq q_2 \leq p_2 + \delta \) and \( m - \delta < p_1, q_1 < m + \delta \).

2. Claim: Each point only needs compared with \( \leq 6 \) other points.
Implementation details

▶ Need to identify which points to compare in $O(n)$ time
▶ Assume points are sorted by $y$-coordinate. Ensure list is passed to each recursion sorted.
▶ Given sorted list, it’s easy to find the relevant points to compare
  1. Remove points whose $x$-coordinate differs from $m$ by more than $\delta$.
  2. Scan through rest from bottom to top, compare each point with the next 6 points in the list.
▶ Can find dividing line that splits $P_L$ and $P_R$ in $O(n)$ time.
Proof of Claim

- All points in to be compared with $p$ lie in a $\delta \times \delta$ rectangle.
- Since each is at least $\delta$ away from the others, we can draw circles of radius $r = \delta/2$ around each and these circles do not overlap.
- The area of the intersection of a circle and the box is at least $\pi r^2 / 4$.
- Since the total area of the rectangle is $\delta^2$, the total number of points must be at most $\delta^2 / (\pi r^2 / 4) = 16/\pi = 5.09\ldots < 6$.
- Better constants are possible but the exact constant isn’t important.