Selling Chocolate

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2. You make $1 profit from Choco and $6 profit from Choco Deluxe
3. Daily demand is 200 bars of Choco and 300 bars of Choco Deluxe
4. Your factory can produce at most 400 bars of chocolate a day
5. To maximize profit, what should you order from the factory?
Selling Chocolate: Linear Program

Let

\[ x_1 = \text{number of bars of Choco ordered} \]
\[ x_2 = \text{number of bars of Choco Deluxe ordered} \]
Selling Chocolate: Linear Program

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Objective:

\[ \max x_1 + 6x_2 \]
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\[ x_1 \leq 200 \]
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Helpful to draw the “feasible region”…
Concepts

Definition
A linear program is *infeasible* if the constraints are so tight that it is impossible to satisfy all of them. E.g., $x \leq 1, x \geq 2$. 
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*If the linear program is feasible and bounded, the optimum is achieved at a vertex of the feasible region.*
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Algorithm (Tedious Algorithm)
*Compute the objective function at each vertex...*
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Theorem
*If the linear program is feasible and bounded, the optimum is achieved at a vertex of the feasible region.*

Algorithm (Tedious Algorithm)
*Compute the objective function at each vertex. . . but this may take exponential time.*
Better Algorithm: Simplex Algorithm

Simplex Algorithm was devised by George Dantzig in 1947...
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Algorithm

*Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.*
Better Algorithm: Simplex Algorithm

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Algorithm

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Not known to be polynomial time but very quick in practice. Polynomial time algorithms do exist but are less used in practice.
Selling Chocolate Again

- You chocolate shop launches a new product “Choco Supreme” that gives $13 profit per bar
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$$\max x_1 + 6x_2 + 13x_3$$
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Need to visualize in 3D...
How do we know that a solution is optimal?

1. Suppose your friend claims that $3100 is the optimum for

$$\max \quad x_1 + 6x_2 + 13x_3$$

and that this is achieved with $x_1 = 0, x_2 = 300, x_3 = 100$. 

2. Revisit constraints to certify that solution if optimal:

   $x_1 \leq 200 (1)$
   
   $x_2 \leq 300 (2)$
   
   $x_1 + x_2 + x_3 \leq 400 (3)$
   
   $x_2 + 3x_3 \leq 600 (4)$

3. Note that $0 \cdot (1) + 1 \cdot (2) + 1 \cdot (3) + 4 \cdot (4)$ is

   $x_1 + 6x_2 + 13x_3 \leq 3100$

4. But how did we come up with the coefficients $(0, 1, 1, 4)$?
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Duality

- Back to simpler example: max \( x_1 + 6x_2 \) subject to

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- Adding one copy of Eq. (1) and seven copies of Eq. (2) gives

\[
x_1 + 7x_2 \leq 2300
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- Adding five copies of Eq. (2) and one copy of Eq. (3) gives
  \[
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  \]
More Duality

1. Trying to find multipliers that give good upper bound:

<table>
<thead>
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3. Finding best such upper bound is new LP!

Minimize: $200y_1 + 300y_2 + 400y_3$

subject to

$$y_1 + y_3 \geq 1, \quad y_2 + y_3 \geq 6, \quad y_1, y_2, y_3 \geq 0$$

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Duality in General

Primal and Dual Linear Programs:

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Let \( \text{opt}_\text{primal} \) be optimal solution of Primal LP and let \( \text{opt}_\text{dual} \) be optimal solution of Dual LP:

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and hence, any feasible solution of the dual LP upper bounds \( \text{opt}_\text{primal} \).
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