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2. You make $1 profit from Choco and $6 profit from Choco Deluxe
3. Daily demand is 200 bars of Choco and 300 bars of Choco Deluxe
4. Your factory can produce at most 400 bars of chocolate a day
5. To maximize profit, what should you order from the factory?
Selling Chocolate: Linear Program

Let

\[ x_1 = \text{number of bars of Choco ordered} \]
\[ x_2 = \text{number of bars of Choco Deluxe ordered} \]
Selling Chocolate: Linear Program

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Helpful to draw the “feasible region”…
Concepts

Definition
A linear program is *infeasible* if the constraints are so tight that it is impossible to satisfy all of them. E.g., $x \leq 1, x \geq 2$. 
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*If the linear program is feasible and bounded, the optimum is achieved at a vertex of the feasible region.*
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Algorithm (Tedious Algorithm)
Compute the objective function at each vertex...
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**Algorithm (Tedious Algorithm)**
*Compute the objective function at each vertex... but this may take exponential time.*
Better Algorithm: Simplex Algorithm

Simplex Algorithm was devised by George Dantzig in 1947...
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Algorithm

*Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.*
Better Algorithm: Simplex Algorithm

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Algorithm

Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.

Not known to be polynomial time but very quick in practice. Polynomial time algorithms do exist but are less used in practice.
Selling Chocolate Again

- You chocolate shop launches a new product “Choco Supreme” that gives $13 profit per bar
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Need to visualize in 3D…
How do we know that a solution is optimal?

1. Suppose your friend claims that $3100$ is the optimum for

$$\max \quad x_1 + 6x_2 + 13x_3$$

and that this is achieved with $x_1 = 0 \quad x_2 = 300 \quad x_3 = 100$. 

2. Revisit constraints to certify that solution if optimal:

- $x_1 \leq 200$ (1)
- $x_2 \leq 300$ (2)
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- $x_2 + 3x_3 \leq 600$ (4)

3. Note that $0 \cdot \text{Eq. (1)} + 1 \cdot \text{Eq. (2)} + 1 \cdot \text{Eq. (3)} + 4 \cdot \text{Eq. (4)}$ is

$$x_1 + 6x_2 + 13x_3 \leq 3100$$

4. But how did we come up with the coefficients $(0, 1, 1, 4)$?
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Duality

▶ Back to simpler example: \( \text{max } x_1 + 6x_2 \) subject to

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- Adding five copies of Eq. (2) and one copy of Eq. (3) gives

\[
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More Duality

1. Trying to find multipliers that give good upper bound:

<table>
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3. Finding best such upper bound is new LP!

Minimize: $200y_1 + 300y_2 + 400y_3$

subject to

$$y_1 + y_3 \geq 1, \quad y_2 + y_3 \geq 6, \quad y_1, y_2, y_3 \geq 0$$
### Duality in General

#### Primal and Dual Linear Programs:

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**Theorem**

Let \( \text{opt}_{\text{primal}} \) be optimal solution of Primal LP and let \( \text{opt}_{\text{dual}} \) be optimal solution of Dual LP:

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and hence, any feasible solution of the dual LP upper bounds \( \text{opt}_{\text{primal}} \).
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