From Last Time: Quicksort

**Problem:** Sort an array of distinct values \( X = [x_1, \ldots, x_n] \)

**Algorithm**

1. *Pick a pivot* \( x \in X \) *at random from the array*
2. *Construct new arrays* \( Y = [y_1, \ldots, y_k], \ Z = [z_1, \ldots, z_{n-k-1}] \) *where*
   \[
   y < x < z \text{ for all } y \in Y, \ z \in Z
   \]
3. *Recursively sort* \( Y \) *and* \( Z \) *to get* \( Y' \) *and* \( Z' \)
4. *Return the array that concatenates* \( Y', x, \) *and* \( Z' \)

What’s the expected number of comparisons performed in this algorithm?
Probability two items are compared

Lemma
Let $a$ and $b$ be the $i$-th and $j$-th smallest element of $X$ where $i < j$.

\[
Pr[a \text{ is compared to } b] = \frac{2}{j - i + 1}
\]

Proof.
1. Consider $S = \{x \in X : a \leq x \leq b\}$
2. $a$ and $b$ are compared iff the first pivot chosen from $S$ is either $a$ or $b$
3. Elements of $S$ are equally likely to be chosen as a pivot, so

\[
Pr[a \text{ is compared to } b] = \frac{2}{|S|} = \frac{2}{j - i + 1}
\]

\[\square\]
Expected Number of Comparisons

Lemma

*Expected number of comparisons performs is $O(n \log n)$.*

Proof.

1. Let $Z_{ij} = 1$ if the $i$-th smallest element is compared to $j$-th smallest element and $Z_{ij} = 0$ otherwise.
2. Number of comparisons: $\sum_{1 \leq i < j \leq n} Z_{ij}$
3. Expected number of comparisons:

   $$E \left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] = \sum_{1 \leq i < j \leq n} E[Z_{ij}] = \sum_{1 \leq i < j \leq n} \frac{2}{j - i + 1} = \sum_{j=2}^{n} \sum_{k=2}^{j} \frac{2}{k}$$

4. Because $H_n = 1 + 1/2 + 1/3 + \ldots + 1/n = O(\log n)$,

   $$E \left[ \sum_{1 \leq i < j \leq n} Z_{ij} \right] \leq \sum_{j=2}^{n} \sum_{k=2}^{j} \frac{2}{k} = n \cdot O(\log n) = O(n \log n)$$
Outline

Karger’s Randomized Min-Cut Algorithm
Min-Cut Problem

Given an unweighted, multi-graph $G = (V, E)$, we want to partition $V$ into $V_1$ and $V_2$ such that $|E \cap (V_1 \times V_2)|$ is minimized.

Algorithm

- Contract a random edge $e = (u, v)$ and remove self-loops but not multi-edges
- Repeat until there are only 2 vertices remaining.
- Output the number of remaining edges.

Let $|V| = n$ and $|E| = m$. 
Example

Step 1

Step 2

Step 3

Step 4

Step 5

Step 6
Correctness with low probability

**Theorem**

*Algorithm is correct with probability* \( \geq \frac{2}{n^2} \) *and never underestimates.*

**Proof.**

- Min cut of the graph doesn’t decrease: after \( e = (x, y) \) contracted, set of possible cuts is limited to all those with \( x \) and \( y \) on same side
- Let \( C = (V_1, V_2) \) be a specific minimum cut with \( |C| = k \).
- Let \( A_i \) be event that we don’t contract edge across \( C \) at step \( i \).

\[
P[\cap_{1 \leq i \leq n-2} A_i] = P[A_1] P[A_2|A_1] \ldots P[A_{n-2}|\cap_{1 \leq i \leq n-3} A_i]
\]

- Number of edges before \( i \)-th step if no edges across \( C \) have been contracted so far is at least \((n - i + 1)k/2\) since there are \( n - i + 1 \) nodes remaining each with degree \( \geq k \)
- \( P[A_i|A_1 \cap A_2 \cap \ldots \cap A_{i-1}] \geq 1 - 2/(n - i + 1) \) and so

\[
P[\cap_{1 \leq i \leq n-2} A_i] \geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right)\ldots\left(1 - \frac{2}{3}\right) = \frac{n-2}{n} \cdot \frac{n-3}{8n-1} \cdot \frac{n-4}{n-2} \cdot \ldots \cdot \frac{1}{3} = \frac{2}{n(n-1)}
\]
Min-Cut Problem: Boosting the probability

**Theorem**

Repeating $\alpha n^2/2$ times (with new random coin flips) and returning smallest cut is correct with probability at least $1 - e^{-\alpha}$.

**Proof.**

- Because each repeat is independent,

  $$\mathbb{P} \left[ \text{always fails} \right] = \prod_{1 \leq i \leq \alpha n^2/2} \mathbb{P} \left[ \text{i-th try fails} \right] \leq \left( 1 - 2/n^2 \right)^{\alpha n^2/2}$$

- Use fact $1 - x \leq e^{-x}$ for $x \geq 0$ and simplify. 

\[ \square \]