Shortest Paths

Let $G = (V, E)$ be a directed graph with weights $w : E \to \mathbb{R}^+$. 

**Definition**

For path $p = (v_1, \ldots, v_k)$ be a path, define

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

The *shortest path* between $u$ and $v$ is

$$\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}$$

if there is a path from $u$ to $v$ and $\infty$ otherwise.
Dijkstra’s Warm-Up

Single-Source Problem: Given $s \in V$, find $\delta(s, v)$ for all $v \in V$.

Dijkstra’s algorithm solves problem if all edges are non-negative:

- Maintains array $(d[v] : v \in V)$ where $d[v]$ will always be $\infty$ or the length of some path from $s$ to $v$, not necessarily the shortest. Hence,

$$d[v] \geq \delta(s, v)$$

- Maintains a set of processed vertices $R$. We’ll prove that for all $v \in R$:

$$d[v] = \delta(s, v)$$
Dijkstra’s Algorithm

Algorithm

1. \(d[s] = 0\) and for \(s \neq v:\)

\[
d[v] = w(s, v) \text{ if } (s, v) \in E \text{ and } \infty \text{ otherwise}
\]

2. \(R \leftarrow \{s\}\)

3. While \(|R| < |V|:\)
   3.1 \(u \leftarrow \arg\min_{v \not\in R} d[v]\)
   3.2 \(R \leftarrow R + u\)
   3.3 For each \(v \not\in R\) that is a neighbor of \(u:\)

\[
d[v] = \min(d[u] + w(u, v), d[v])
\]

Running Time: \(O(|V|^2)\) for simple implementation but can be improved.
1. Step 1: $d[s] = 0$, $d[a] = 3$, $d[b] = 6$, $d[c] = \infty$, and $R = \{s\}$

2. Step 2: $d[s] = 0$, $d[a] = 3$, $d[b] = 5$, $d[c] = 12$, and $R = \{s, a\}$

3. Step 3: $d[s] = 0$, $d[a] = 3$, $d[b] = 5$, $d[c] = 8$, and $R = \{s, a, b\}$

4. Step 4: $d[s] = 0$, $d[a] = 3$, $d[b] = 5$, $d[c] = 8$, and $R = \{s, a, b, c\}$
Correctness of Algorithm

The correctness of the algorithm follows because a) $d[v]$ never increases, b) $d[v] \geq \delta(s, u)$ at all times, and c) appealing to the following lemma:

**Lemma**

*When $u$ is added to $R$, $d[u] = \delta(s, u)$*
When \( u \) gets added to \( R \), \( d[u] \) is correct (1/2)

Let \( d_u[v] \) be value of \( d[v] \) just before \( u \) is chosen as minimum.

**Lemma**

*For all* \( u \), \( d_u[u] = \delta(s, u) \)

- By contradiction: Let \( u \) be first vertex put in \( R \) with \( d_u[u] > \delta(s, u) \)
- Consider a shortest path from \( s \) to \( u \). Let \( y \) be first vertex not in \( R \). Note that \( y \) may or may not be \( u \).

- Claim: \( d_u[y] = \delta(s, y) \)
  - Let \( x \) be the predecessor of \( y \) on the path. Note that \( x \in R \).
  - \( d_x[x] = \delta(s, x) \) by assumption that \( u \) is first bad vertex.
  - After iteration where \( x \) is added to \( R \): \( d[y] \leq \delta(s, x) + w(x, y) \)
  - \( \delta(s, x) + w(x, y) = \delta(s, y) \) since path included shortest path to \( y \)
When $u$ gets added to $R$, $d[u]$ is correct (2/2)

Let $d_u[v]$ be value of $d[v]$ just before $u$ is chosen as minimum.

Lemma

For all $u$, $d_u[u] = \delta(s, u)$

- By contradiction: Let $u$ be first vertex put in $R$ with $d_u[u] > \delta(s, u)$
- Consider a shortest path from $s$ to $u$. Let $y$ be first vertex not in $R$. Note that $y$ may or may not be $u$.

- Claim: $d_u[y] = \delta(s, y)$
- Since $y$ lies on shortest path to $u$: $\delta(s, y) \leq \delta(s, u)$
- Putting above two lines together:

$$d_u[y] = \delta(s, y) \leq \delta(s, u) < d_u[u]$$

- If $y \neq u$: Contradiction because $u$ was the next minimum and so $d_u[u] \leq d_u[y]$
- If $y = u$: Contradicts $d_u[y] < d_u[u]"