CMPSCI 611: Advanced Algorithms
Lecture 7: Bipartite Matchings and Union Find

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Outline

Intersection of Matroids and Bipartite Matchings

Union-Find Data Structure
Problem

- Input: Bipartite graph $B = (U, V, E)$ where $U, V$ are disjoint sets of vertices and $E$ is a set of edges between $U$ and $V$.
- Output: The matching (i.e., subset of $E$ where no two edges share a vertex) of maximum size.
Bipartite Matchings

Problem

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Example Application: There’s a set of tasks $V$ to be performed and a set of individuals $U$, each capable of doing a subset of the tasks.

- Each person may be assigned to at most one task.
- At most one person may be assigned a task.
- Not every person can do every task. Can encode this in $E$. 
Intersection of Matroids

- The bipartite matching subset system is not a matroid but it is the intersection of two matroids.
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Define:

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\[ I' = \text{subsets of } E \text{ where each } v \in V \text{ has degree at most 1} \]

Then, bipartite matching subset system is \((E, I \cap I')\)
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Then, bipartite matching subset system is \((E, I \cap I')\)

**Theorem**

*For matroids \((E, I)\) and \((E, I')\), the largest set in \(I \cap I'\) can be found in time \(O(|E|^3 \cdot C(I, I'))\) where \(C(I, I')\) is time to check \(i \in I\) or \(i \in I'\).*

We won’t prove this general theorem but will focus on the special case of bipartite matching. Note that there is no analogous theorem for the intersection of three matroids.
Augmenting Paths Definitions

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An *augmenting path* is an odd sequence of edges that begins and ends at (different) free vertices and alternates between matching edges $e \in M$ and non-matching edges $e \in E - M$. 
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Definition
An augmenting path is an odd sequence of edges that begins and ends at (different) free vertices and alternates between matching edges $e \in M$ and non-matching edges $e \in E - M$.

Definition
If $P$ is an augmenting path for matching $M$, the symmetric difference of $M$ and $P$ is $M \oplus P := (M \cup P) - (M \cap P)$. 
Augmenting Paths Properties

Lemma

For matching $M$ and augmenting path $P$, $M \oplus P$ is a matching and

$$|M \oplus P| = |M| + 1.$$
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Algorithm: Find augmenting paths until we can’t find anymore!
Finding an augmenting path allows us to “grow” matching

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Proof.

- A matching is a graph where no node has degree $> 1$
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- Size of matching is (number of degree 1 nodes)/2
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- Remove edges in $P \cap M$ and add edges in $P \setminus M$:
  - Adds one to degree of two nodes that initially were free.
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- Remove edges in $P \cap M$ and add edges in $P \setminus M$:
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  - Degree of interior points of $P$ still have degree 1.
Augmenting path exists for non-maximum matching

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Proof.
- Let $M'$ be a matching such that $|M'| > |M|$
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**Lemma**

*If* $M$ *is non-maximum matching, there exists an augmenting path.*

**Proof.**

- Let $M'$ be a matching such that $|M'| > |M|$
- Consider $E' = M \oplus M'\ldots$
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**Lemma**

*If $M$ is non-maximum matching, there exists an augmenting path.*

**Proof.**

- Let $M'$ be a matching such that $|M'| > |M|
- Consider $E' = M \oplus M'$...consists of simple paths and cycles whose edges alternate between $M$ and $M'$
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- Cycles have the same number of edges from $M'$ and $M$
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- Cycles have the same number of edges from $M'$ and $M$
- There must exist a path $P$ with more edges from $M'$ than $M$, i.e., one that starts and end with an edge in $M'$
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- Cycles have the same number of edges from $M'$ and $M$
- There must exist a path $P$ with more edges from $M'$ than $M$, i.e., one that starts and end with an edge in $M'$
- This is an augmenting path: edges alternate between $M'$ and $M$ and it starts and ends with free vertices
Bipartite Matching Algorithm

Algorithm

- $M \leftarrow \emptyset$
- While there exists an augmenting path $P$: $M \leftarrow M \oplus P$
- Return $M$
Bipartite Matching Algorithm

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We can find an augmenting path in \( O(|U||E|) \) time:

- Direct matched edges \( V \rightarrow U \) and non-matched edges \( U \rightarrow V \)
- For each free vertex \( u \in U \), grow a BFS: If a free vertex \( v \in V \) is reachable from \( u \), we have an augmenting path
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Total running time is $O(\min(|U|, |V|)|U||E|)$ because the maximum matching size is at most $\min(|U|, |V|)$. Can be improved by finding the augmenting paths in a more clever way.
Outline

Intersection of Matroids and Bipartite Matchings

Union-Find Data Structure
Recall Kruskal’s Algorithm... 

**Problem:** Given an undirected, connected graph $G = (V, E)$ with positive edge weights, find the minimum-weight subset $E' \subset E$ such that the graph $G = (V, E')$ is a minimum spanning tree.
Recall Kruskal’s Algorithm...

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**Algorithm (Kruskal)**

1. *Sort edges by non-decreasing weight*
2. $F = \emptyset$
3. Until $F$ is a spanning tree of $G$
   - 3.1 *Get the next edge* $e$
   - 3.2 *If* $F + e$ *is acyclic then* $F = F + e$
Recall Kruskal’s Algorithm . . .

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We saw how to implement this with \( O(|E| \log |E| + |V|^2) \) running time. This class: improving to \( O(|E| \log |E|) \) via the union-find data structure.
Union-Find Data Structure

Encodes a set of disjoint sets where each set contains an element designated as the “label” of the set. E.g.,

\{a, b, c\} labeled “a” \{d, e, f\} labeled “e”
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Supports three operations:

1. Make-Set(\(v\)): Adds a set \(\{v\}\) with label “\(v\)"
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2. Union-Set(\textit{u}, \textit{v}): Replaces sets including \textit{u} and \textit{v} with a new set that is union of both sets and labels this set by some element it contains.
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2. **Union-Set(\(u, v\))**: Replaces sets including \(u\) and \(v\) with a new set that is union of both sets and labels this set by some element it contains. For example, the result of Union-Set(\(f, v\)) is

   \{a, b, c\} labeled “a” \{d, e, f, v\} labeled “e”

3. **Find(\(v\))**: Returns the label of the set including \(v\)
Kruskal’s Algorithm with Union-Find

Algorithm (Kruskal)

1. Sort edges by non-decreasing weight
2. For each vertex $v \in V$: Make-Set($v$)
3. $F = \emptyset$
4. For each edge $e = (u, v)$ in $E$
   4.1 If $\text{Find}(u) \neq \text{Find}(v)$ then $\text{Union}(u, v)$ and $F = F + e$
Kruskal’s Algorithm with Union-Find

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Well, how should we implement union-find...
Simple Implementation of Union-Find

1. Each disjoint set is stored as a linked list of nodes.
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   - Update “next” pointer at end of longer list to point to start of shorter list
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   - Update “label” pointers of shorter list to point to label of other list
   - Update auxiliary pointers and size information
Theorem
Consider a sequence of $m$ operations including $n$ Make-Set operations. Total running time is $O(m + n \log n)$. 
Union-Find Analysis

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    \end{itemize}
\end{itemize}

Hence, Kruskal’s algorithm can be implemented in time

\[ \begin{align*}
O(|E| \log |E|) + O(|E| + |V| \log |V|) &= O(|E| \log |E|)
\end{align*} \]
Faster Implementation of Union Find

Theorem

There exists an implementation that, given a sequence of $n$ Make-Set operations and $m$ total operations, takes $O(m\alpha(n))$ time where $\alpha$ is the inverse Ackermann’s function.
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There exists an implementation that, given a sequence of \( n \) Make-Set operations and \( m \) total operations, takes \( O(m\alpha(n)) \) time where \( \alpha \) is the inverse Ackermann’s function.

Definition (Ackermann’s Function)
Define a sequence of functions: \( A_0(x) = 1 + x \) and

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A_k(x) = A_{k-1}(A_{k-1}(\ldots A_{k-1}(x)\ldots))
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where \( A_{k-1} \) is applied \( x \) times. Ackermann function is \( A(k) = A_k(2) \).
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where $A_{k-1}$ is applied $x$ times. Ackermann function is $A(k) = A_k(2)$. $\alpha(n)$ is defined as smallest $k$ such that $A(k) \geq n$. 
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where \( A_{k-1} \) is applied \( x \) times. Ackermann function is \( A(k) = A_k(2) \). \( \alpha(n) \) is defined as smallest \( k \) such that \( A(k) \geq n \).

**Example**
\( \alpha(n) \leq 4 \) for all \( n \leq 2^{2^{2^{\ldots 2^{2048}}}} \) where tower is of height 2048.
Idea Behind Faster Implementation

- Store each set as a rooted tree.

1. Make-Set($v$): Takes $O(1)$ time to add a single node.
2. Find($v$): Takes $O(d_v)$ time where $d_v$ is the depth of $v$.
3. Union-Set($u$, $v$): $O(d_v + d_u)$ time
   - Perform Find($u$) and Find($v$).
   - Add pointer from root of smaller tree to root of larger tree.
   - Extra Trick! Do path compression. When we do a Find operation, update the pointers from all nodes encountered to point to the root. Increases time by a constant factor but saves time for future Find operations.

More Details: See Section 21.4 of CLRS (3rd edition) or Section 5.1 of DPV.
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3. **Union-Set(u, v)**: $O(d_v + d_u)$ time
Idea Behind Faster Implementation

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**More Details:** See Section 21.4 of CLRS (3rd edition) or Section 5.1 of DPV.