Problem: Suppose $A(x)$ and $B(x)$ are polynomials of degree $n - 1$:

\[
A(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}
\]

\[
B(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-1} x^{n-1}
\]

Compute $C(x) = A(x)B(x)$. We’ll assume $n$ is a power of 2.

How long does naive algorithm take? $O(n^2)$
Representation of Polynomials

Definition
The *coefficient representation* (CR) of a polynomial the vector of coefficients. E.g., \((1, 3, -2, 1)\) is the coefficient representation of

\[ f(x) = 1 + 3x - 2x^2 + x^3 \]

Definition
The *point-value representation* (PVR) of a polynomial: for \(n\) distinct points \(x_0, \ldots, x_{n-1}\) the PVR of \(f\) is

\[ \{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_{n-1}, f(x_{n-1}))\} \]

E.g., \(f(x) \equiv \{(0, 1), (1, 3), (2, 7), (3, 19)\}\).

Lemma
*Specifying the value of a function at \(n\) distinct points uniquely specifies a degree \(n - 1\) polynomial that goes through those points.*
First attempt: Let $x_0, \ldots, x_{n-1}$ be distinct and suppose

$$A(x) \equiv \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$$

$$B(x) \equiv \{(x_0, z_0), (x_1, z_1), \ldots, (x_{n-1}, z_{n-1})\}$$

Then surely,

$$C(x) \equiv \{(x_0, y_0z_0), (x_1, y_1z_1), \ldots, (x_{n-1}, y_{n-1}z_{n-1})\}$$

Issue: While $C(x_i) = y_iz_i$, $C$ is a degree $2n - 2$ polynomial and we need $2n - 1$ distinct points to specify it.

Fix: Assume $A$ and $B$ are specified on at least $2n - 1$ distinct points.

Can compute PVR of $C$ is $\Theta(n)$ time. But what about coefficient representation?
Framework for Fast Polynomial Multiplication

- Input: Coefficient representation of $A(x)$ and $B(x)$
- Step 1: Transform into PVR by evaluating on at least $2n - 1$ points
- Step 2: Multiply polynomials to get $C(x)$ in PVR
- Step 3: Transform PVR of $C(x)$ back into CR.

Naive implementation of step 1 takes $O(n^2)$ time. We’ll do steps 1 and 3 in $O(n \log n)$ time.

Important: We can choose any distinct points for the PVR. Let’s use the complex roots of unity...
Complex Roots of Unity

Definition
The $n$-th roots of unity are the complex solutions to the equation $x^n = 1$, i.e.,
\[ e^{2\pi ik/n} = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n} \quad k = 0, \ldots, n - 1. \]

Let $\omega_n = e^{2\pi i/n}$.

Lemma (Halving Lemma)
The squares of the $2n$-th roots of unity are two copies of the $n$-th roots of unity:
\[ \{(\omega_{2n}^0)^2, \ldots, (\omega_{2n}^{2n-1})^2\} = \{\omega_n^0, \ldots, \omega_n^{n-1}\} \cup \{\omega_n^0, \ldots, \omega_n^{n-1}\} \]

Proof.
Follows since $(\omega_{2n}^r)^2 = e^{2r \cdot 2\pi i/(2n)} = e^{r \cdot 2\pi i/n} = \omega_n^r$ and $(\omega_{2n}^{r+n})^2 = \omega_n^r$. □
Divide and Conquer for Polynomial Evaluation

- Write degree $n - 1$ polynomial to be evaluated in terms of two degree $n/2 - 1$ polynomials:

  \[
  A(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}
  = (a_0 + a_2 x^2 + \ldots + a_{n-2} x^{n-2})
  + x(a_1 + a_3 x^2 + \ldots + a_{n-1} x^{n-2})
  = A_{\text{even}}(x^2) + xA_{\text{odd}}(x^2)
  \]

- To evaluate $A$ at $2n$-th roots of unity, we evaluate $A_{\text{even}}$ and $A_{\text{odd}}$ at $x^2$ for

  \[x \in \{\omega_0^2, \omega_1^2, \ldots, \omega_{2n}^{2n-1}\}\]

- If $T(n)$ is time to evaluate degree $n - 1$ poly at $2n$-th roots of unity,

  \[T(1) = \Theta(1) \quad \text{and} \quad T(n) = 2T(n/2) + \Theta(n)\]

- Use Master Theorem to conclude that $T(n) = \Theta(n \log n)$. 

Input: Coefficient representation of $A(x)$ and $B(x)$

Step 1: Transform into PVR by evaluating at at least $2n - 1$ points

Step 2: Multiply polynomials to get $C(x)$ in PVR

Step 3: Transform PVR of $C(x)$ back into CR.

We now know:

1. Step 1 can be done in $O(n \log n)$ time.
2. Step 2 can be done in $O(n)$ time.

It turns out that Step 3 is almost identical to Step 1!
**Polynomial Evaluation and Interpolation**

**Step 1 Revisited:** Transform \((a_0, a_1, \ldots, a_{n-1})\) to

\[
\{(\omega_{2n}^0, y_0), (\omega_{2n}^1, y_1), \ldots, (\omega_{2n}^{2n-1}, y_{2n-1})\}
\]

where \(y_i = A(\omega_{2n}^i)\). In other words, we need to evaluate:

\[
V_n \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{2n-1} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{2n-1} \end{pmatrix}
\]

where \(a_i = 0\) for \(i \geq n - 1\) and

\[
V_n = \begin{pmatrix}
1 & 1 & 1 & 1 & \ldots & 1 \\
1 & \omega_{2n} & \omega_{2n}^2 & \omega_{2n}^3 & \ldots & \omega_{2n}^{2n-1} \\
1 & \omega_{2n}^2 & \omega_{2n}^4 & \omega_{2n}^6 & \ldots & \omega_{2n}^{2(2n-1)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega_{2n}^{2n-1} & \omega_{2n}^{2(2n-1)} & \omega_{2n}^{3(2n-1)} & \ldots & \omega_{2n}^{(2n-1)(2n-1)}
\end{pmatrix}
\]
Polynomial Evaluation and Interpolation

Step 3 as inverse of Step 1: Need to transform

\[ \{(\omega_{2n}^0, y_0), (\omega_{2n}^1, y_1), \ldots, (\omega_{2n}^{2n-1}, y_{2n-1})\} \]

into \((a_0, a_1, \ldots, a_{2n-1})\) where \(y_i = A(\omega_{2n}^i)\). In other words, we need

\[
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_{2n-1}
\end{pmatrix}
= V_{2n}^{-1} \cdot
\begin{pmatrix}
y_0 \\
y_1 \\
y_2 \\
\vdots \\
y_{2n-1}
\end{pmatrix}
\]

The inverse of \(V_{2n}\) is just \(V_{2n}\) with \(\omega_{2n}\) replaced by \(\omega_{2n}^{-1}\)

\[
V_{2n}^{-1} = \frac{1}{2n}
\begin{pmatrix}
1 & 1 & 1 & 1 & \ldots & 1 \\
1 & \omega_{2n}^{-1} & \omega_{2n}^{-2} & \omega_{2n}^{-3} & \ldots & \omega_{2n}^{-(2n-1)} \\
1 & \omega_{2n}^{-2} & \omega_{2n}^{-4} & \omega_{2n}^{-6} & \ldots & \omega_{2n}^{-(2n-1)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega_{2n}^{-(2n-1)} & \omega_{2n}^{-(2n-1)} & \omega_{2n}^{-(2n-1)} & \ldots & \omega_{2n}^{-(2n-1)(2n-1)}
\end{pmatrix}
\]
Solving Step 3 Outline

- Need to compute:
  \[ a_k = \frac{\hat{A}(\omega_{2n}^{-k})}{2n} \text{ for } k = 0, \ldots, 2n - 1 \]

  where \( \hat{A}(x) = y_0 + y_1x + \ldots + y_{2n-1}x^{2n-1} \)

- Rewrite \( \hat{A}(x) = \hat{A}_{\text{even}}(x^2) + x\hat{A}_{\text{odd}}(x^2) \)

- To evaluate \( \hat{A} \) on

  \[ \{\omega_{2n}^0, \omega_{2n}^{-1}, \ldots, \omega_{2n}^{-(2n-1)}\} \]

  it suffices to evaluate \( \hat{A}_{\text{even}} \) and \( \hat{A}_{\text{odd}} \) on

  \[ \{\omega_{n}^0, \omega_{n}^{-1}, \ldots, \omega_{n}^{-(n-1)}\} \]

  because Halving Lemma also applies to \( \omega_{2n}^{-1} \).

- Step 3 can also be done in \( O(n \log n) \) steps.