Selling Chocolate

1. You run up a chocolate shop that sells “Choco” and ”Choco Deluxe”
2. You make $1 profit from Choco and $6 profit from Choco Deluxe
3. Daily demand is 200 bars of Choco and 300 bars of Choco Deluxe
4. Your factory can produce at most 400 bars of chocolate a day
5. To maximize profit, what should you order from the factory?
Selling Chocolate: Linear Program

Let

\[ x_1 = \text{number of bars of Choco ordered} \]
\[ x_2 = \text{number of bars of Choco Deluxe ordered} \]

Objective:

\[ \max x_1 + 6x_2 \]

Constraints:

\[ x_1 \leq 200 \]
\[ x_2 \leq 300 \]
\[ x_1 + x_2 \leq 400 \]
\[ x_1, x_2 \geq 0 \]

Helpful to draw the “feasible region” …
Concepts

Definition
A linear program is infeasible if the constraints are so tight that it is impossible to satisfy all of them. E.g., $x \leq 1, x \geq 2$.

Definition
A linear program is unbounded if the constraints are so loose that it is possible to achieve arbitrarily high objective values. E.g., $\max x_1 + x_2$ subject to $x_1, x_2 \geq 0$.

Theorem
If the linear program is feasible and bounded, the optimum is achieved at a vertex of the feasible region.

Algorithm (Tedious Algorithm)
Compute the objective function at each vertex... but this may take exponential time.
Better Algorithm: Simplex Algorithm

Simplex Algorithm was devised by George Dantzig in 1947.

Algorithm

Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.

Not known to be polynomial time but very quick in practice. Polynomial time algorithms do exist but are less used in practice.
Selling Chocolate Again

- You chocolate shop launches a new product “Choco Supreme” that gives $13 profit per bar
- Let $x_3$ be the number of bars of Supreme manufactured
- Deluxe and Supreme use same packaging machine: $x_2 + 3x_3 \leq 600$

Objective:

$$\max x_1 + 6x_2 + 13x_3$$

Constraints:

- $x_1 \leq 200$
- $x_2 \leq 300$
- $x_1 + x_2 + x_3 \leq 400$
- $x_2 + 3x_3 \leq 600$
- $x_1, x_2, x_3 \geq 0$

Need to visualize in 3D…
How do we know that a solution is optimal?

1. Suppose your friend claims that $3100 is the optimum for

$$\text{max } x_1 + 6x_2 + 13x_3$$

and that this is achieved with $x_1 = 0$, $x_2 = 300$, $x_3 = 100$.

2. Revisit constraints to certify that solution if optimal:

$$x_1 \leq 200 \tag{1}$$

$$x_2 \leq 300 \tag{2}$$

$$x_1 + x_2 + x_3 \leq 400 \tag{3}$$

$$x_2 + 3x_3 \leq 600 \tag{4}$$

3. Note that $0 \cdot \text{Eq. (1)} + 1 \cdot \text{Eq. (2)} + 1 \cdot \text{Eq. (3)} + 4 \cdot \text{Eq. (4)}$ is

$$x_1 + 6x_2 + 13x_3 \leq 3100$$

4. But how did we come up with the coefficients (0, 1, 1, 4)?
Back to simpler example: max $x_1 + 6x_2$ subject to

\[
\begin{align*}
    x_1 & \leq 200 \\
    x_2 & \leq 300 \\
    x_1 + x_2 & \leq 400 \\
    x_1, x_2 & \geq 0
\end{align*}
\]

Claim that optimal solution has value 1900 where $x_1 = 100$, $x_2 = 300$

Adding one copy of Eq. (1) and seven copies of Eq. (2) gives

\[
x_1 + 7x_2 \leq 2300
\]

and so $x_1 + 6x_2 \leq 2300$ because $x_1, x_2 \geq 0$

Adding five copies of Eq. (2) and one copy of Eq. (3) gives

\[
x_1 + 6x_2 \leq 1900
\]
More Duality

1. Trying to find multipliers that give good upper bound:

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 200$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 300$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + x_2 \leq 400$</td>
</tr>
</tbody>
</table>

gives inequality $(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$.

2. If $y_1 + y_3 \geq 1$, $y_2 + y_3 \geq 6$, $y_1, y_2, y_3 \geq 0$, then an upper bound is

$$200y_1 + 300y_2 + 400y_3$$

3. Finding best such upper bound is new LP!

Minimize: $200y_1 + 300y_2 + 400y_3$

subject to

$$y_1 + y_3 \geq 1, \quad y_2 + y_3 \geq 6, \quad y_1, y_2, y_3 \geq 0$$
**Duality in General**

Primal and Dual Linear Programs:

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \max c^T x )</td>
<td>( \min y^T b )</td>
</tr>
<tr>
<td>( Ax \leq b )</td>
<td>( y^T A \geq c^T )</td>
</tr>
<tr>
<td>( x \geq 0 )</td>
<td>( y \geq 0 )</td>
</tr>
</tbody>
</table>

**Theorem**

Let \( \text{OPT}_{\text{primal}} \) be optimal solution of Primal LP and let \( \text{OPT}_{\text{dual}} \) be optimal solution of Dual LP:

\[
\text{OPT}_{\text{primal}} = \text{OPT}_{\text{dual}}
\]

and hence, any feasible solution of the dual LP upper bounds \( \text{OPT}_{\text{primal}} \).