Selling Chocolate

1. You run up a chocolate shop that sells “Choco” and “Choco Deluxe”
2. You make $1 profit from Choco and $6 profit from Choco Deluxe
3. Daily demand is 200 bars of Choco and 300 bars of Choco Deluxe
4. Your factory can produce at most 400 bars of chocolate a day
5. To maximize profit, what should you order from the factory?
Selling Chocolate: Linear Program

Let

\[ x_1 = \text{number of bars of Choco ordered} \]
\[ x_2 = \text{number of bars of Choco Deluxe ordered} \]

Objective:

\[ \text{max } x_1 + 6x_2 \]

Constraints:

\[ x_1 \leq 200 \]
\[ x_2 \leq 300 \]
\[ x_1 + x_2 \leq 400 \]
\[ x_1, x_2 \geq 0 \]

Helpful to draw the “feasible region”…
Concepts

**Definition**
A linear program is *infeasible* if the constraints are so tight that it is impossible to satisfy all of them. E.g., \( x \leq 1, x \geq 2 \).

**Definition**
A linear program is *unbounded* if the constraints are so loose that it is possible to achieve arbitrarily high objective values. E.g., \( \max x_1 + x_2 \) subject to \( x_1, x_2 \geq 0 \).

**Theorem**
*If the linear program is feasible and bounded, the optimum is achieved at a vertex of the feasible region.*

**Algorithm (Tedious Algorithm)**
*Compute the objective function at each vertex... but this may take exponential time.*
Better Algorithm: Simplex Algorithm

Simplex Algorithm was devised by George Dantzig in 1947...

Algorithm

*Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.*

Not known to be polynomial time but very quick in practice. Polynomial time algorithms do exist but are less used in practice.
You chocolate shop launches a new product “Choco Supreme” that gives $13 profit per bar

Let $x_3$ be the number of bars of Supreme manufactured

Deluxe and Supreme use same packaging machine: $x_2 + 3x_3 \leq 600$

Objective:

$$\max x_1 + 6x_2 + 13x_3$$

Constraints:

$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 + x_3 \leq 400$$

$$x_2 + 3x_3 \leq 600$$

$$x_1, x_2, x_3 \geq 0$$

Need to visualize in 3D...
How do we know that a solution is optimal?

1. Suppose your friend claims that $3100 is the optimum for

\[ \max \ x_1 + 6x_2 + 13x_3 \]

and that this is achieved with \( x_1 = 0, x_2 = 300, x_3 = 100 \).

2. Revisit constraints to certify that solution if optimal:

\[
\begin{align*}
x_1 & \leq 200 \quad (1) \\
x_2 & \leq 300 \quad (2) \\
x_1 + x_2 + x_3 & \leq 400 \quad (3) \\
x_2 + 3x_3 & \leq 600 \quad (4)
\end{align*}
\]

3. Note that \(0 \cdot \text{Eq. (1)} + 1 \cdot \text{Eq. (2)} + 1 \cdot \text{Eq. (3)} + 4 \cdot \text{Eq. (4)} \) is

\[ x_1 + 6x_2 + 13x_3 \leq 3100 \]

4. But how did we come up with the coefficients \((0, 1, 1, 4)\)?
Duality

- Back to simpler example: max $x_1 + 6x_2$ subject to
  
  $x_1 \leq 200$
  
  $x_2 \leq 300$
  
  $x_1 + x_2 \leq 400$
  
  $x_1, x_2 \geq 0$

- Claim that optimal solution has value 1900 where $x_1 = 100$, $x_2 = 300$

- Adding one copy of Eq. (1) and seven copies of Eq. (2) gives
  
  $x_1 + 7x_2 \leq 2300$
  
  and so $x_1 + 6x_2 \leq 2300$ because $x_1, x_2 \geq 0$

- Adding five copies of Eq. (2) and one copy of Eq. (3) gives
  
  $x_1 + 6x_2 \leq 1900$
More Duality

1. Trying to find multipliers that give good upper bound:

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 200$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 300$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + x_2 \leq 400$</td>
</tr>
</tbody>
</table>

gives inequality $(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$.

2. If $y_1 + y_3 \geq 1$, $y_2 + y_3 \geq 6$, $y_1, y_2, y_3 \geq 0$, then an upper bound is

$$200y_1 + 300y_2 + 400y_3$$

3. Finding best such upper bound is new LP!

Minimize: $200y_1 + 300y_2 + 400y_3$

subject to

$$y_1 + y_3 \geq 1, \quad y_2 + y_3 \geq 6, \quad y_1, y_2, y_3 \geq 0$$
Duality in General

Primal and Dual Linear Programs:

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max c^T x$</td>
<td>$\min y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c^T$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Theorem

Let $\text{OPT}_{\text{primal}}$ be optimal solution of Primal LP and let $\text{OPT}_{\text{dual}}$ be optimal solution of Dual LP:

$$\text{OPT}_{\text{primal}} = \text{OPT}_{\text{dual}}$$

and hence, any feasible solution of the dual LP upper bounds $\text{OPT}_{\text{primal}}$. 