CMPSCI 611: Advanced Algorithms
Lecture 21: Metric Traveling Salesperson

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Outline

Metric TSP 2-approx

Metric TSP 3/2 approximate
Metric Traveling Salesperson Problem

- **Input**: Weighted complete graph $G = (V, E)$ with positive weights such that for edges $e = (u, v)$, $e' = (v, w)$, and $e'' = (u, w)$

  \[ w_e + w_{e'} \geq w_{e''} \]

- **Goal**: Find the tour (a path that visits every node exactly once and returns to starting point) of minimum total weight.
Metric TSP Approximation Algorithm

Algorithm
1. Compute minimum spanning tree $T_{mst}$ of $G$
2. Consider a “pseudo-tour” that walks around $T_{mst}$
3. Create tour from pseudo-tour by skipping pre-visited nodes

Theorem
The algorithm is a 2-approximation.

Proof.
- Cost of pseudo-tour is twice cost of $T_{mst}$
- Cost of tour found is at most cost of pseudo-tour:

$$\text{cost(tour found)} \leq \text{cost(pseudo tour)} = 2 \cdot \text{cost}(T_{mst})$$

- Cost of $T_{mst}$ is at most cost of optimal tour since removing an edge in an optimal tour gives a spanning tree:

$$\text{cost}(T_{mst}) \leq \text{cost(optimal tour)}$$
Outline

Metric TSP 2-approx

Metric TSP 3/2 approximate
**Eulerian Tours**

**Definition**
A Eulerian tour is a path that traverses every edge of a graph exactly once and returns back to the initial vertex.

**Lemma**
A graph contains an Eulerian tour iff $G$ is connected and every vertex has even degree.
Algorithm

1. Compute minimum spanning tree $T_{\text{mst}}$ of $G$
2. Let $D$ be the nodes in $T_{\text{mst}}$ that have odd degree
3. Find minimum cost perfect matching $M$ on nodes of $D$
4. Find Euler tour of $T_{\text{mst}} + M$
5. Transform into tour by short-cutting repeated vertices.
Theorem

*The algorithm is a 3/2-approximation and runs in polynomial time.*

Proof.

- Cost of tour found is at most cost of Euler tour
  \[
  \text{cost(tour found)} \leq \text{cost(Euler tour)} = \text{cost}(T_{\text{mst}}) + \text{cost}(M)
  \]

- As before, \( \text{cost}(T_{\text{mst}}) \leq \text{cost(optimal tour)} \)

- Cost of \( M \) is at most half cost of optimal tour
  \[
  \text{cost}(M) \leq \frac{1}{2} \text{cost(optimal tour)}
  \]

Let \( D = \{d_1, \ldots, d_k\} \) be ordered according to optimal tour.

\[
\text{cost(optimal tour)} \geq \sum_{i=1}^{k-1} w_{d_i,d_{i+1}} + w_{d_k,d_1} = \left( \sum_{i=2}^{k} w_{d_i,d_{i+1}} \right) + \left( w_{d_2,d_3} + w_{d_4,d_5} + \cdots + w_{d_k,d_1} \right)
\]