Definitions

Input:
- Directed Graph $G = (V, E)$
- Capacities $C(u, v) > 0$ for $(u, v) \in E$ and $C(u, v) = 0$ for $(u, v) \notin E$
- A source node $s$, and sink node $t$
Capacity
Definitions

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Output: A flow $f$ from $s$ to $t$ where $f : V \times V \to \mathbb{R}$ satisfies
- Skew-symmetry: $\forall u, v \in V, f(u, v) = -f(v, u)$
- Conservation of Flow: $\forall v \in V - \{s, t\}, \sum_{u \in V} f(u, v) = 0$
- Capacity Constraints: $\forall u, v \in V, f(u, v) \leq C(u, v)$

Goal: Maximize “size of the flow”, i.e., the total flow coming leaving $s$:

$$|f| = \sum_{v \in V} f(s, v)$$
Capacity

![Graph with vertices and edges labeled with capacities such as s to v3 with capacity 13, v1 to v3 with capacity 16, and v2 to t with capacity 20. Edges also have capacities such as v1 to v2 with 12 and v3 to v4 with 14.]
Capacity/Flow

\[ \begin{align*}
    v_1 & \quad v_2 \\
    s & \rightarrow & \quad 16/11 & \rightarrow & \quad 12/12 & \rightarrow & \quad v_3 \\
    v_1 & \rightarrow & \quad 10/0 & \rightarrow & \quad 4/1 & \rightarrow & \quad v_4 \\
    v_1 & \rightarrow & \quad 13/8 & \rightarrow & \quad v_3 \\
    v_2 & \rightarrow & \quad 20/15 & \rightarrow & \quad v_4 \\
    v_3 & \rightarrow & \quad 9/4 & \rightarrow & \quad v_2 \\
    v_4 & \rightarrow & \quad 7/7 & \rightarrow & \quad v_2 \\
    v_4 & \rightarrow & \quad 14/11 & \rightarrow & \quad v_3 \\
    v_4 & \rightarrow & \quad 4/4 & \rightarrow & \quad t
\end{align*} \]
Cut Definitions

Definition
An $s - t$ cut of $G$ is a partition of the vertices into two sets $A$ and $B$ such that $s \in A$ and $t \in B$.

Definition
The capacity of a cut $(A, B)$ is

$$C(A, B) = \sum_{u \in A, v \in B} C(u, v)$$

Definition
The flow across a cut $(A, B)$ is

$$f(A, B) = \sum_{u \in A, v \in B} f(u, v)$$

Note that because of capacity constraints: $f(A, B) \leq C(A, B)$
First Cut

The diagram shows a network with nodes labeled $s$, $v_1$, $v_2$, $v_3$, $v_4$, and $t$. The edges between the nodes are labeled with fractions representing the weight or cost of the edge. The values are as follows:

- $s$ to $v_1$: $16/11$
- $v_1$ to $v_2$: $12/12$
- $v_1$ to $v_3$: $10/0$
- $v_2$ to $v_4$: $20/15$
- $v_2$ to $t$: $4/4$
- $v_3$ to $v_4$: $14/11$
- $v_3$ to $t$: $7/7$
- $v_4$ to $t$: $4/4$

The network has a specific path from $s$ to $t$.

8/20
Second Cut

Diagram with nodes labeled as follows:
- **s**: Source node
- **v1**: Intermediate node
- **v2**: Intermediate node
- **v3**: Intermediate node
- **v4**: Intermediate node
- **t**: Terminal node

Edges with weights:
- s → v1: 16/11
- s → v3: 13/8
- v1 → v2: 12/12
- v1 → v3: 4/1
- v2 → v4: 20/15
- v3 → v2: 9/4
- v4 → t: 4/4
- v3 → v4: 14/11
- v3 → v1: 10/0
- v4 → t: 7/7

The diagram illustrates a network with weighted edges and nodes.
Lemma
For any flow $f$: for all s-t cuts $(A, B)$, $f(A, B)$ equals $|f|$.

Theorem (Max-Flow Min-Cut)
For any flow network and flow $f$, the following statements are equivalent:

1. $f$ is a maximum flow.
2. There exists an s – t cut $(A, B)$ such that $|f| = C(A, B)$

We’ll prove both next class.
Residual Networks and Augmenting Paths

Residual network encodes how you can change the flow between two nodes given the current flow and the capacity constraints.

Definition
Given a flow network $G = (V, E)$ and flow $f$ in $G$, the residual network $G_f$ is defined as

$$G_f = (V, E_f) \text{ where } E_f = \{(u, v) : C(u, v) - f(u, v) > 0\}$$

$$C_f(u, v) = C(u, v) - f(u, v)$$

Note that $(u, v) \in E_f$ implies either $C(u, v) > 0$ or $C(v, u) > 0$.

Definition
An augmenting path for flow $f$ is a path from $s$ to $t$ in graph $G_f$. The bottleneck capacity $b(p)$ is the minimum capacity in $G_f$ of any edge of $p$. We can increase flow by $b(p)$ along an augmenting path.
Capacity/Flow

Graph:
- Nodes: $s$, $v_1$, $v_2$, $v_3$, $v_4$, $t$
- Edges:
  - $s$ to $v_1$: $16/11$
  - $s$ to $v_3$: $13/8$
  - $v_1$ to $v_2$: $12/12$
  - $v_3$ to $v_4$: $9/4$
  - $v_3$ to $v_1$: $4/1$
  - $v_4$ to $v_2$: $7/7$
  - $v_4$ to $t$: $4/4$
  - $v_2$ to $t$: $20/15$
  - $v_1$ to $v_3$: $10/0$
  - $v_2$ to $v_3$: $14/11$

Flow/Capacity:
- $s$ to $v_1$: $16/11$
- $v_1$ to $v_2$: $12/12$
- $v_2$ to $t$: $20/15$
- $v_3$ to $v_4$: $9/4$
- $v_4$ to $t$: $4/4$
- $s$ to $v_3$: $13/8$
- $v_3$ to $v_1$: $4/1$
- $v_3$ to $v_4$: $14/11$
- $v_1$ to $v_3$: $10/0$
- $v_2$ to $v_3$: $7/7$

Total Flow:
- $s$ to $t$: $16/11 + 12/12 + 20/15 = 48/39$
- Total Capacity:
  - $s$ to $v_1$: $16/11$
  - $v_1$ to $v_2$: $12/12$
  - $v_2$ to $t$: $20/15$
  - $v_3$ to $v_4$: $9/4$
  - $v_4$ to $t$: $4/4$
  - $s$ to $v_3$: $13/8$
  - $v_3$ to $v_1$: $4/1$
  - $v_3$ to $v_4$: $14/11$
  - $v_1$ to $v_3$: $10/0$
  - $v_2$ to $v_3$: $7/7$

Flow Constraints:
- Each edge flow must be less than or equal to its capacity.
- Flow conservation at nodes: $\sum_{in} x_{ij} - \sum_{out} x_{ij} = 0$ for all nodes except source and sink.

Flow Analysis:
- Flow from $s$ to $t$ is within capacity constraints.
- Flow network is analyzed for maximum flow possible.

Network Flow Theorem:
- The maximum flow from source $s$ to sink $t$ is equal to the minimum cut capacity in the network.

Conclusion:
- Maximum flow possible is determined.
- Flow network optimized for maximum throughput.
Residual

\[
\begin{align*}
\text{Residual} & \quad v_1 & \quad v_2 \\
& & \quad s & \quad t
\end{align*}
\]
Augmenting Path

\begin{figure}
\centering
\includegraphics[width=\textwidth]{augmenting_path.png}
\caption{Example of an augmenting path in a network.}
\end{figure}
Old Flow

\[ s \rightarrow v_3 \rightarrow v_2 \rightarrow t \]
\[ s \rightarrow v_3 \rightarrow v_1 \rightarrow v_2 \rightarrow t \]
\[ s \rightarrow v_4 \rightarrow t \]
\[ s \rightarrow v_3 \rightarrow v_4 \rightarrow t \]
New Flow

Graph with nodes labeled $s$, $v_1$, $v_2$, $v_3$, and $v_4$. Edges and capacities are indicated as follows:

- $s$ to $v_1$: 16/11
- $s$ to $v_3$: 13/12
- $v_1$ to $v_2$: 12/12
- $v_1$ to $v_3$: 10/0
- $v_2$ to $t$: 20/19
- $v_2$ to $v_4$: 7/7
- $v_3$ to $v_4$: 14/11
- $v_4$ to $t$: 4/4

Nodes $v_1$ and $v_2$ are connected by a direct edge with capacity 12/12.
Min Capacity Cut Proves this is Optimal
Old Residual Graph
New Residual Graph
Ford-Fulkerson Algorithm

Algorithm

1. \( \text{flow } f = 0 \)
2. \( \text{while there exists an augmenting path } p \text{ for } f \)
   2.1 find augmenting path \( p \)
   2.2 augment \( f \) by \( b(p) \) units along \( p \)
3. return \( f \)

Theorem

The algorithms finds a maximum flow in time \( O(|E||f^*|) \) if capacities are integral where \( |f^*| \) is the size of the maximum flow.

Proof.

\( O(|E|) \) time to find each augmenting path via BFS and \( |f^*| \) iterations because each augmenting path increases flow by at least 1.