Shortest Paths

Let $G = (V, E)$ be a directed graph with weights $w : E \rightarrow \mathbb{R}^+$.  

**Definition**  
For path $p = (v_1, \ldots, v_k)$ be a path, define  

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

The *shortest path* between $u$ and $v$ is  

$$\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}$$

if there is a path from $u$ to $v$ and $\infty$ otherwise.
Dijkstra’s Warm-Up

Single-Source Problem: Given $s \in V$, find $\delta(s, v)$ for all $v \in V$.

Dijkstra’s algorithm solves problem if all edges are non-negative:

- Maintains array $(d[v] : v \in V)$ where $d[v]$ will always be $\infty$ or the length of some path from $s$ to $v$, not necessarily the shortest. Hence,

\[ d[v] \geq \delta(s, v) \]

- Maintains a set of processed vertices $R$. We’ll prove that for all $v \in R$:

\[ d[v] = \delta(s, v) \]
Dijkstra’s Algorithm

Algorithm

1. \(d[s] = 0\) and for \(s \neq v\):

\[
d[v] = w(s, v) \text{ if } (s, v) \in E \text{ and } \infty \text{ otherwise}
\]

2. \(R \leftarrow \{s\}\)

3. While \(|R| < |V|\):
   3.1 \(u \leftarrow \arg\min_{v \notin R} d[v]\)
   3.2 \(R \leftarrow R + u\)
   3.3 For each \(v \notin R\) that is a neighbor of \(u\):

\[
d[v] = \min(d[u] + w(u, v), d[v])
\]

Running Time: \(O(|V|^2)\) for simple implementation but can be improved.
Correctness of Algorithm

The correctness of the algorithm follows because a) \( d[v] \) never increases, b) \( d[v] \geq \delta(s, u) \) at all times, and c) appealing to the following lemma:

**Lemma**

*When \( u \) is added to \( R \), \( d[u] = \delta(s, u) \)*
When $u$ gets added to $R$, $d[u]$ is correct

Let $d_u[v]$ be value of $d[v]$ just before $u$ is chosen as minimum.

**Lemma**

*For all* $u$, $d_u[u] = \delta(s, u)$

- **By contradiction:** Let $u$ be first vertex put in $R$ with $d_u[u] > \delta(s, u)$
- Consider a shortest path from $s$ to $u$. Let $y$ be first vertex not in $R$. Note that $y$ may or may not be $u$.
- **Claim:** $d_u[y] = \delta(s, y)$
  - Let $x$ be the predecessor of $y$ on the path. Note that $x \in R$.
  - $d_x[x] = \delta(s, x)$ by assumption that $u$ is first bad vertex.
  - After iteration where $x$ is added to $R$: $d[y] \leq \delta(s, x) + w(x, y)$
  - $\delta(s, x) + w(x, y) = \delta(s, y)$ since path included shortest path to $y$
- Since $y$ lies on shortest path to $u$: $\delta(s, y) \leq \delta(s, u)$
- Putting above two lines together:
  $$d_u[y] = \delta(s, y) \leq \delta(s, u) < d_u[u]$$
- **If** $y \neq u$: Contradiction because $u$ was the next minimum and so
  $$d_u[u] \leq d_u[y]$$
- **If** $y = u$: Contradiction because we deduced above $d_u[y] = \delta(s, y)$