Outline

Summary of Matroid Results
The Problem

Definition
A subset system $S = (E, \mathcal{I})$ is a finite set $E$ with a collection $\mathcal{I}$ of subsets of $E$ such that if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$.

Problem Given a subset system $S = (E, \mathcal{I})$ and weight function $w : E \to \mathbb{R}^+$, find $A \in \mathcal{I}$ such that $w(A) = \sum_{e \in A} w(e)$ is maximized.

Algorithm (Greedy)

1. $A = \emptyset$
2. Sort elements of $E$ by non-increasing weight
3. For each $e \in E$: If $A + e \in \mathcal{I}$ then $A \leftarrow A + e$
Matroid Definition and Theorem

Definition
Subset system \((E, \mathcal{I})\) has the **exchange property** if

\[
\forall A, B \in \mathcal{I} : (|A| < |B|) \implies (\exists e \in B - A \text{ such that } A + e \in \mathcal{I})
\]

Theorem
Given a subset system \((E, \mathcal{I})\), the following statements are equivalent:

1. Greedy algorithm returns optimal solution for any weight function.
2. The subset system obeys the exchange property.
3. The subset system obeys the cardinality property.

where we say \(A \in \mathcal{I}\) is a maximal subset of \(E'\) if \(A \subseteq E'\) and there doesn't exist \(e \in E'\) such that \(A + e \in \mathcal{I}\).
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A subset system \((E, \mathcal{I})\) has the cardinality property if

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\forall E' \subseteq E : (A, B \in \mathcal{I} \text{ maximal subsets of } E') \implies (|A| = |B|)
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Suppose $A, B$ are maximal subsets of $E' \subseteq E$. Need to show $|A| = |B|$
Exchange Property implies Cardinality Property

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- If $|B| > |A|$, the exchange property implies

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- Note that $A + e$ would still be in $E'$ since $e \in B \subseteq E'$. 
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  $$\exists \ e \in B - A \text{ such that } A + e \in I$$

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- Thus $A$ was not maximal in $E'$. Contradiction!
Cardinality Property implies Exchange Property

- Suffices to show that \((E, \mathcal{I})\) not a matroid implies there exists \(E'\) and \(A, B \in \mathcal{I}\) such that \(|A| \neq |B|\) and \(A, B\) are maximal in \(E'\)
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- \((E, \mathcal{I})\) not a matroid implies that

\[
\exists A, C \in \mathcal{I} \text{ such that } |A| < |C| \text{ and } \not\exists e \in C - A \text{ with } A + e \in \mathcal{I}
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- Define \(E' = A \cup C\) and note that \(A\) is maximal in \(E'\).
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Cardinality Property implies Exchange Property

- Suffices to show that $(E, I)$ not a matroid implies there exists $E'$ and $A, B \in I$ such that $|A| \neq |B|$ and $A, B$ are maximal in $E'$.
- $(E, I)$ not a matroid implies that
  \[ \exists A, C \in I \text{ such that } |A| < |C| \text{ and } \forall e \in C - A \text{ with } A + e \in I \]
- Define $E' = A \cup C$ and note that $A$ is maximal in $E'$.
- There exists $B \in I$ such that $C \subseteq B$ and $B$ is maximal in $E'$.
- But $|B| \geq |A| + 1$ as required.
Example 1

**Theorem**

The Maximum Weight Forest (MWF) subset system is a matroid.
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Proof.

1. Pick an arbitrary subset of edges $E' \subseteq E$.
2. Let $n_1, \ldots, n_k$ be the number of nodes in the connected components.
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Proof.

- Pick an arbitrary subset of edges $E' \subseteq E$.
- Let $n_1, \ldots, n_k$ be the number of nodes in the connected components.
- Any maximal acyclic subset of $E'$ has size

\[(n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) = n - k\]

because a maximal acyclic subgraph of a connected graph on $n_i$ nodes is a tree and has $n_i - 1$ edges.
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because a maximal acyclic subgraph of a connected graph on $n_i$ nodes is a tree and has $n_i - 1$ edges.
▶ Cardinality Theorem implies that it’s a matroid.
Example 2

Theorem

Let $E$ be a set of directed edges and $\mathcal{I}$ be subsets such that no two edges in the same subset point to same node. This is a matroid.

Proof.

For any $E' \subseteq E$, the number of edges in a maximal subset of $E'$ is equal to the number of vertices pointed to in $E'$. The Cardinality Theorem implies that it's a matroid.
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