CMPSCI 611: Advanced Algorithms
Lecture 6: Cardinality Theorem and Matroid Examples

Andrew McGregor
Outline

Summary of Matroid Results
The Problem

Definition
A subset system $S = (E, \mathcal{I})$ is a finite set $E$ with a collection $\mathcal{I}$ of subsets of $E$ such that if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$.

Problem Given a subset system $S = (E, \mathcal{I})$ and weight function $w : E \to \mathbb{R}^+$, find $A \in \mathcal{I}$ such that $w(A) = \sum_{e \in A} w(e)$ is maximized.

Algorithm (Greedy)

1. $A = \emptyset$
2. Sort elements of $E$ by non-increasing weight
3. For each $e \in E$: If $A + e \in \mathcal{I}$ then $A \leftarrow A + e$
Matroid Definition and Theorem

Definition
Subset system \((E, \mathcal{I})\) has the exchange property if

\[
\forall A, B \in \mathcal{I} : (|A| < |B|) \implies (\exists e \in B - A \text{ such that } A + e \in \mathcal{I})
\]

Definition
A subset system \((E, \mathcal{I})\) has the cardinality property if

\[
\forall E' \subseteq E : (A, B \in \mathcal{I} \text{ maximal subsets of } E') \implies (|A| = |B|)
\]

where we say \(A \in \mathcal{I}\) is a maximal subset of \(E'\) if \(A \subseteq E'\) and there doesn’t exist \(e \in E'\) such that \(A + e \in \mathcal{I}\).

Theorem
Given a subset system \((E, \mathcal{I})\), the following statements are equivalent:

1. Greedy algorithm returns optimal solution for any weight function.
2. The subset system obeys the exchange property.
3. The subset system obeys the cardinality property.
Suppose $A, B$ are maximal subsets of $E' \subseteq E$. Need to show $|A| = |B|$

If $|B| > |A|$, the exchange property implies

$$\exists \ e \in B - A \text{ such that } A + e \in \mathcal{I}$$

Note that $A + e$ would still be in $E'$ since $e \in B \subseteq E'$.

Thus $A$ was not maximal in $E'$. Contradiction!
Cardinality Property implies Exchange Property

- Suffices to show that \((E, \mathcal{I})\) not a matroid implies there exists \(E'\) and \(A, B \in \mathcal{I}\) such that \(|A| \neq |B|\) and \(A, B\) are maximal in \(E'\)
- \((E, \mathcal{I})\) not a matroid implies that
  \[
  \exists A, C \in \mathcal{I} \text{ such that } |A| < |C| \text{ and } \forall e \in C - A \text{ with } A + e \in \mathcal{I}
  \]
- Define \(E' = A \cup C\) and note that \(A\) is maximal in \(E'\).
- There exists \(B \in \mathcal{I}\) such that \(C \subseteq B\) and \(B\) is maximal in \(E'\).
- But \(|B| \geq |A| + 1\) as required.
Example 1

Theorem

The Maximum Weight Forest (MWF) subset system is a matroid.

Proof.

- Pick an arbitrary subset of edges $E' \subseteq E$.
- Let $n_1, \ldots, n_k$ be the number of nodes in the connected components.
- Any maximal acyclic subset of $E'$ has size

$$ (n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) = n - k $$

because a maximal acyclic subgraph of a connected graph on $n_i$ nodes is a tree and has $n_i - 1$ edges.
- Cardinality Theorem implies that it’s a matroid.
Example 2

Theorem
Let $E$ be a set of directed edges and $I$ be subsets such that no two edges in the same subset point to the same node. This is a matroid.

Proof.
- For any $E' \subseteq E$, the number of edges in a maximal subset of $E'$ is equal to the number of vertices pointed to in $E'$.
- Cardinality Theorem implies that it’s a matroid.