Question 1. Let $G$ be an undirected graph $G = (V, E)$ with $n$ nodes and $m$ edges.

(1) Let $S$ be a random subset of the nodes, i.e., each node is independently added to $S$ with probability $1/2$. What is the expected value and variance of the size of the cut $(S, V \setminus S)$?

(2) Prove that it is possible to partition the nodes of $G$ into four groups such that at least $3m/4$ of the edges are cut, i.e., the endpoints are in different groups.

Question 2. Let $S$ be a set of $n$ distinct values. We say value $x$ is an $\epsilon$-approximate median if the rank of $x$ is at least $n(1/2 - \epsilon)$ and at most $n(1/2 + \epsilon)$. We say $x$ is an $\epsilon$-approximate mean if $\mu(1 - \epsilon) \leq x \leq \mu(1 + \epsilon)$ where $\mu$ is the mean of the values in $S$.

Consider sampling each element of $S$ with probability $p$ and let $R$ be the resulting subset of $S$.

(1) Prove a lower bound (the higher the better) on $p$ such that the median of $R$ is an $\epsilon$-approximate median of $S$ with probability at least 99/100.

(2) Prove a lower bound (the higher the better) on $p$ such that the mean of $R$ is an $\epsilon$-approximate mean of $S$ with probability at least 99/100.

(3) Repeat part two of the question on the assumption that $\mu = 1$ and every value in $S$ is at least 0 and at most 2.

Question 3. Consider the greedy algorithm for weighted matching, i.e., consider the edges in decreasing order of weight and add an edge to the current solution $M$ if it does not share an endpoint with an edge already in the current solution. Let $w_{\text{alg}}$ be the total weight of the edges chosen by the greedy algorithm and let $w_{\text{opt}}$ be the maximum weight of a matching in the graph.

Prove an upper bound (the lower the better) on the ratio $w_{\text{opt}}/w_{\text{alg}}$.

Question 4. Toss $n$ totally independent coins and let $X_i = 1$ if the $i$-th coin is heads and $X_i = 0$ otherwise. For each of the $2^n$ subsets $S \subset \{1, \ldots, n\}$, define $Y_S = (\sum_{i \in S} X_i) \mod 2$. Let $Y = \sum_{S \subset \{1, \ldots, n\}} Y_S$ and compute the exact values for $E[Y]$ and $\mathbb{V}[Y]$. Prove the best upper bound you can on $\mathbb{P}[|Y - 2^{n-1}| \geq t]$ as a function of $t$. Update: You can answer the question as written, but if you prefer, you can replace the last part by bounding $\mathbb{P}[|Y - 2^{n-1} + 1/2| \geq t]$.

Question 5. Let $A[1], \ldots, A[2k]$ be a list of values that are sorted in increasing order. Your goal is find the $i \in [k]$ such that $f(i) = A[i+k] - A[i]$ is maximized. Unfortunately you don’t have time to test all $k$ pairs but luckily you are happy if you return $i$ such that $f(i) \geq \alpha \max_j f(j)$ for some specified $\alpha < 1$.

(1) If $\alpha = 1/2$, design an algorithm that only requires reading $O(1)$ values from the list.

(2) If $\alpha = 1 - \epsilon$, design an algorithm that only requires reading $O(\epsilon^{-1} \log k)$ values from the list.
**Hint:** For the first part consider reading $A[1], A[k], A[k + 1]$, and $A[2k]$. For the second part, one possible approach involves performing $O(1/\epsilon)$ binary searches. Remember to analyze the running time and prove the approximation ratio.