Homework may be completed in group of size with at most four students. You're not allowed to use material from the web or talk about the homework with anybody outside your collaboration group (aside from the lecturer or TA.)

- Solutions should be typed and uploaded as a pdf to gradescope.com.
- To get full marks, answers must be sufficiently detailed, supported with rigorous proofs (of both correctness and running time analysis). Faster algorithms will typically get more marks than slower algorithms.

**Question 1.** Let $G$ be an undirected graph $G = (V, E)$ with $n$ nodes and $m$ edges.

1. Let $S$ be a random subset of the nodes, i.e., each node is independently added to $S$ with probability 1/2. What is the expected value and variance of the size of the cut $(S, V \setminus S)$?
2. Prove that it is possible to partition the nodes of $G$ into four groups such that at least $3m/4$ of the edges are cut, i.e., the endpoints are in different groups.

**Question 2.** Let $S$ be a set of $n$ distinct values. We say value $x$ is an $\epsilon$-approximate median if the rank of $x$ is at least $n(1/2 - \epsilon)$ and at most $n(1/2 + \epsilon)$. We say $x$ is an $\epsilon$-approximate mean if $\mu(1 - \epsilon) \leq x \leq \mu(1 + \epsilon)$ where $\mu$ is the mean of the values in $S$. Consider sampling each element of $S$ with probability $p$ and let $R$ be the resulting subset of $S$.

1. Prove a lower bound (the higher the better) on $p$ such that the median of $R$ is an $\epsilon$-approximate median of $S$ with probability at least $99/100$.
2. Prove a lower bound (the higher the better) on $p$ such that the mean of $R$ is an $\epsilon$-approximate mean of $S$ with probability at least $99/100$.
3. Repeat part two of the question on the assumption that $\mu = 1$ and every value in $S$ is at least 0 and at most 2.

**Question 3.** Consider the greedy algorithm for weighted matching, i.e., consider the edges in decreasing order of weight and add an edge to the current solution $M$ if it does not share an endpoint with an edge already in the current solution. Let $w_{\text{alg}}$ be the total weight of the edges chosen by the greedy algorithm and let $w_{\text{opt}}$ be the maximum weight of a matching in the graph. Prove an upper bound (the lower the better) on the ratio $w_{\text{opt}}/w_{\text{alg}}$.

**Question 4.** TBD.

**Question 5.** TBD.