CMPSCI 611: “Advanced Algorithms”
Lecture 24: More Approximation Algorithms and Review

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Outline

More Approximation Algorithms

Divide and Conquer

Greedy Algorithms

Dynamic Programming and Shortest Paths

Network Flows

Randomized Algorithms

NP Completeness

Approximation Algorithms

Linear Programming
Formulating Vertex Cover as a Linear (?) Program

- Given graph $G = (V, E)$, for each node $v \in V$, create variable $x_v$
- For each edge $(u, v) \in E$, create constraint $x_v + x_u \geq 1$

Minimize $\sum_{v \in V} x_v$ subject to

- $x_v + x_u \geq 1$ for all $(u, v) \in E$
- $x_v \leq 1$ for all $v \in V$
- $x_v \geq 0$ for all $v \in V$

Does this mean we can solve Vertex Cover in poly-time? No, need to constraints $x_v \in \{0, 1\}$ and program is linear integer program.
LP Relaxation

- Vertex cover can be expressed as the following integer program
- Minimize $\sum_{v \in V} x_v$ subject to
  \[ x_v + x_u \geq 1 \quad \text{for all } (u, v) \in E \]
  \[ x_v \leq 1 \quad \text{for all } v \in V \]
  \[ x_v \geq 0 \quad \text{for all } v \in V \]

  where each $x_v \in \{0, 1\}$.
- **Relax**: Replace $x_v \in \{0, 1\}$ constraint by $0 \leq x_v \leq 1$
- **Solve**: Let $\hat{x}_v$ be optimal solution.
- **Round**: Let $x'_v = 1$ if $\hat{x}_v \geq 1/2$ and 0 otherwise.
- **Final solution** is feasible for the original ILP and is a 2-approx.
Approx Algorithms and Reductions: Cautionary Tale!

Suppose $\Pi' \leq_P \Pi$ and we have an polynomial time $\alpha$-approximation for a $\Pi$, do we necessarily have an $\alpha$ approximation for $\Pi$?

Problem: Independent Set

- Input: An undirected graph $G = (V, E)$.
- Output: A set $U \subset V$ of maximum size such that no two vertices in $U$ are connected by a single edge.

Lemma

$\text{Independent-Set} \leq_P \text{Vertex-Cover}$

Proof.

$U \subset V$ is an independent set iff $V - U$ is a vertex cover. □

But using a factor 2-approximation for Vertex-Cover may give a factor $\Omega(n)$ approximation for Independent-Set.
Approximation Preserving Reduction

**Problem:** Max-3-SAT

- **Input:** A 3-CNF formula with \( m \) clauses and \( n \) variables
- **Output:** Maximum number of clauses that can be satisfied.

*Recall reduction from 3-SAT:* Given formula \( \phi \), construct graph \( G_\phi \) such that each clique corresponds to a set of simultaneously satisfiable clauses. Hence an \( \alpha \) approx. for CLIQUE gives an \( \alpha \) approx. for MAX-3-SAT.

**Theorem**

*Unless P=NP, for all \( \epsilon > 0 \), there’s no \( (8/7 - \epsilon) \) approx. for Max-3-SAT.*

**Corollary**

*Unless P=NP, for all \( \epsilon > 0 \), there’s no \( (8/7 - \epsilon) \) approx. for CLIQUE.*

(What’s an easy randomized 8/7-approximation for Max-3-SAT?)
Summary of Approximation Algorithms

- Algorithms:
  - 2-approximation for vertex cover
  - 2-approximation for max-cut
  - $3/2$-approximation for metric traveling salesperson
  - $O(\log n)$-approximation for weighted set-cover
  - FPTAS for knapsack

- A poly-time reduction may not be “approximation preserving”

- For a reference of what approximation factors are known check out:
Alternative Approaches to NP-hard problems

- Restrict the input:
  - Finding a clique in graph that is acyclic, of bounded degree, or planar
  - Solving metric TSP where the points are in Euclidean space
- Assume a probability distribution over input: *Average case analysis*
- Assume all integers in the input are polynomial in the input size…

**Definition**
An algorithm runs in *pseudo-polynomial time* if the running time is polynomial in the input size and any integer in the input.

**Definition**
A problem is *strongly NP-complete* if it remains NP-complete even when all integers in an input of length $n$ are polynomial in $n$. 
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Divide and Conquer Methodology

- **Goal:** Solve problem $P$ on an instance $I$ of “size” $n$.

- **Divide & Conquer Method:**
  - Transform $I$ into smaller instances $I_1, \ldots, I_a$ each of “size” $n/b$
  - Solve problem $P$ on each of $I_1, \ldots, I_a$ by recursion
  - Combine the solutions to get a solution of $I$

- **Examples:** Merge Sort, Strassen’s Algorithm, Minimum Distance, Fourier Transform.
Analyzing Divide and Conquer Algorithms

Let $T(n)$ be running time of algorithm on instance of size $n$. Then

$$T(1) = \Theta(1), \quad T(n) = aT(n/b) + \Theta(n^\alpha)$$

where $\Theta(n^\alpha)$ is time to make new instances and combine solutions.

**Theorem (Master Theorem)**

If $a, b, \alpha$ are constants, for $\beta = \log_b a$,

$$T(n) = \begin{cases} 
\Theta(n^\alpha) & \text{if } \alpha > \beta \\
\Theta(n^\beta) & \text{if } \alpha < \beta \\
\Theta(n^\alpha \log n) & \text{if } \alpha = \beta 
\end{cases}$$
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Divide and Conquer

**Greedy Algorithms**

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Generic Problem and Greedy Algorithms

Definition
A *subset system* \( S = (E, \mathcal{I}) \) is a finite set \( E \) with a collection \( \mathcal{I} \) of subsets \( E \) such that:

\[
\text{if } i \in \mathcal{I} \text{ and } i' \subset i \text{ then } i' \in \mathcal{I}
\]

i.e., “\( \mathcal{I} \) is closed under inclusion”

Problem Given a subset system \( S = (E, \mathcal{I}) \) and weight function \( w : E \to \mathbb{R}^+ \), find \( i \in \mathcal{I} \) such that \( w(i) = \sum_{e \in i} w(e) \) is maximized.

Algorithm (Greedy)

1. \( i = \emptyset \)
2. Sort elements of \( E \) by non-increasing weight
3. For each \( e \in E \): If \( i + e \in \mathcal{I} \) then \( i = i + e \)
Matroid Definition and Theorem

Definition
A matroid is a subset system $M = (E, I)$ that satisfies the exchange property: if $i, i' \in I$ such that $|i| < |i'|$, then there exists $e \in i' - i$ with $i + e \in I$

Theorem
For any subset system $(E, I)$, the greedy algorithm solves the optimization problem for $(E, I)$ if and only if $(E, I)$ is a matroid.

- A matroid can also be characterized by the cardinality theorem.
- Maximum bipartite matching can be expressed as intersection of two matroids and can therefore be solved in polynomial time.
- Solving the intersection of three matroids becomes NP-hard.
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Linear Programming
Dynamic Programming and Shortest Paths

When to use dynamic programming...

- **Optimal Substructure**: The solution to the problem can be found using solutions to smaller sub-problems.
- **Overlap of Sub-Problems**: By taking advantage of the fact that many identical sub-problems are created, a dynamic programming algorithm may be more efficient than a divide and conquer algorithm.

Shortest path algorithms...

- **Floyd-Warshall Algorithm**: $O(|V|^3)$
- **Dijkstra’s Algorithm**: Positive weights! $O(|E| + |V| \log |V|)$.
- **Seidel’s Algorithm**: Unweighted Graphs! $O(|V|^2.38)$ running time.
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Definitions

Input:
- Directed Graph $G = (V, E)$
- Capacities $C(u, v) > 0$ for $(u, v) \in E$ and $C(u, v) = 0$ for $(u, v) \notin E$
- A source node $s$, and sink node $t$

Output: A flow $f$ from $s$ to $t$ where $f : V \times V \rightarrow \mathbb{R}$ satisfies
- Skew-symmetry: $\forall u, v \in V, f(u, v) = -f(v, u)$
- Conservation of Flow: $\forall v \in V - \{s, t\}, \sum_{u \in V} f(u, v) = 0$
- Capacity Constraints: $\forall u, v \in V, f(u, v) \leq C(u, v)$

Goal: Maximize “size of the flow”, i.e., the total flow coming leaving $s$:

$$|f| = \sum_{v \in V} f(s, v)$$
Capacity
Cut Definitions

Definition
An \( s - t \) cut of \( G \) is a partition of the vertices into two sets \( A \) and \( B \) such that \( s \in A \) and \( t \in B \).

Definition
The capacity of a cut \((A, B)\) is \( C(A, B) = \sum_{u \in A, v \in B} C(u, v) \)

Definition
The flow across a cut \((A, B)\) is \( f(A, B) = \sum_{u \in A, v \in B} f(u, v) \)

Theorem (Max-Flow Min-Cut)
For any flow network and flow \( f \), the following statements are equivalent:

1. \( f \) is a maximum flow.
2. There exists an \( s - t \) cut \((A, B)\) such that \( |f| = C(A, B) \)

Went over Ford-Fulkerson Algorithm with Edmonds-Karp Heuristic to find max-flow.
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Probability and Examples

- For arbitrary events $A$ and $B$,

\[ \mathbb{P}[A \text{ and } B] = \mathbb{P}[A \text{ given } B] \mathbb{P}[B] \]

and $A$ and $B$ are independent if $\mathbb{P}[A \text{ and } B] = \mathbb{P}[A] \mathbb{P}[B]$.

- Union Bound: $\mathbb{P}[A \text{ or } B] \leq \mathbb{P}[A] + \mathbb{P}[B]$.

- Expectation: $\mathbb{E}[X] = \sum_r r \mathbb{P}[X = r]$.

- Linearity of expectation: $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

- Variance random variable: $\mathbb{V}[X] = \sigma_X^2 = \mathbb{E}[(X - \mathbb{E}[X])^2]$.

- Linearity of variance if $X$ and $Y$ are independent:

\[ \mathbb{V}[X + Y] = \mathbb{V}[X] + \mathbb{V}[Y] \]

Examples: Quicksort, Karger’s Randomized Min-Cut Algorithm, Schwartz-Zippel, Lazy Select, Balls and Bins...
Tail Bounds

**Theorem (Markov)**

Let $Y$ be a non-negative random variable and let $\mu_Y = \mathbb{E}[Y]$. Then, for all $t > 0$, $\mathbb{P}[Y \geq t\mu_Y] \leq 1/t$.

**Theorem (Chebyshev)**

Let $X$ be a random variable with expectation $\mu_X$ and standard deviation $\sigma_X$. Then for $t > 0$, $\mathbb{P}[|X - \mu_X| \geq t\sigma_X] \leq 1/t^2$.

**Theorem**

Let $X_1, \ldots, X_n$ be independent boolean random variables such that $\mathbb{P}[X_i = 1] = p_i$. Then, for $X = \sum_i X_i$, $\mu = \mathbb{E}[X]$, and $\delta > 0$,

$$\mathbb{P}[X > (1 + \delta)\mu] < \left[\frac{e^{\delta}}{(1 + \delta)^{1+\delta}}\right]^{\mu}$$
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NP Completeness

1. $P$: Problems for which there exists a poly-time algorithm
2. $NP$: Problems for which there exists a poly-time algorithm taking advice:
   ▶ If the answer should be “yes”, then there exists advice that leads the algorithm to output “yes”
   ▶ If the answer is “no”, then there doesn’t exist advice that would lead the algorithm to output “yes”
3. A problem $\Pi$ is NP-hard if for any $\Pi' \in NP$: $\Pi' \leq_P \Pi$
4. A problem $\Pi$ is NP-complete if $\Pi \in NP$ and $\Pi$ is NP-hard

Theorem

Clique, vertex cover, subset-sum etc. are NP-Complete.
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Approximation Ratios

Definition
The performance ratio of an algorithm is

\[
\max_{x: |x|=n} \frac{C_{alg}(x)}{C_{opt}(x)}
\]

for a minimization problem

\[
\max_{x: |x|=n} \frac{C_{opt}(x)}{C_{alg}(x)}
\]

for a maximization problem

where \(C_{alg}(x)\) is the value of the algorithm solution on input \(x\) and \(C_{opt}(x)\) is the value of the optimal solution on input \(x\).

Definition
A problem has a PTAS iff for all \(\epsilon > 0\) it has a poly time \((1 + \epsilon)\) approx. A problem has a FPTAS iff for all \(\epsilon > 0\) it has \((1 + \epsilon)\) approx where the run time is poly in \(1/\epsilon\) and poly in the size of the input.

Examples: 2 approx for vertex cover, 2 approx for max-cut, 1.5 approx for metric TSP, \(O(\log n)\)-approx for weighted set-cover
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Linear Programming
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Primal and Dual Linear Programs:

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{max } c^T x$</td>
<td>$\text{min } y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c^T$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

**Theorem**

Let $\text{OPT}_{\text{primal}}$ be optimal solution of Primal LP and let $\text{OPT}_{\text{dual}}$ be optimal solution of Dual LP: If both are bounded and feasible,

$$\text{OPT}_{\text{primal}} = \text{OPT}_{\text{dual}}$$

and hence, any feasible solution of the dual LP upper bounds $\text{OPT}_{\text{primal}}$.

Can be solved in polynomial time but adding integral constraints makes the problem NP-hard.
And finally. . .

Good luck with the exam!