Selling Chocolate

1. You run up a chocolate shop that sells “Choco” and ”Choco Deluxe”
2. You make $1 profit from Choco and $6 profit from Choco Deluxe
3. Daily demand is 200 bars of Choco and 300 bars of Choco Deluxe
4. Your factory can produce at most 400 bars of chocolate a day
5. To maximize profit, what should you order from the factory?
Selling Chocolate: Linear Program

Let

\[ x_1 = \text{number of bars of Choco ordered} \]
\[ x_2 = \text{number of bars of Choco Deluxe ordered} \]

Objective:

\[ \max x_1 + 6x_2 \]

Constraints:

\[ x_1 \leq 200 \]
\[ x_2 \leq 300 \]
\[ x_1 + x_2 \leq 400 \]
\[ x_1, x_2 \geq 0 \]

Helpful to draw the “feasible region”…
Concepts

Definition
A linear program is *infeasible* if the constraints are so tight that it is impossible to satisfy all of them. E.g., $x \leq 1, x \geq 2$.

Definition
A linear program is *unbounded* if the constraints are so loose that it is possible to achieve arbitrarily high objective values. E.g., $\max x_1 + x_2$ subject to $x_1, x_2 \geq 0$.

Theorem
*If the linear program is feasible and bounded, the optimum is achieved at a vertex of the feasible region.*

Algorithm (Tedious Algorithm)
*Compute the objective function at each vertex. . . but this may take exponential time.*
Better Algorithm: Simplex Algorithm

Simplex Algorithm was devised by George Dantzig in 1947…

Algorithm

*Pick arbitrary vertex of the feasible region. Move to adjacent vertex with better objective value. If no such vertex exists, terminate.*

Not known to be polynomial time but very quick in practice. Polynomial time algorithms do exist but are less used in practice.
Selling Chocolate Again

▶ You chocolate shop launches a new product “Choco Supreme” that gives $13 profit per bar
▶ Let $x_3$ be the number of bars of Supreme manufactured
▶ Deluxe and Supreme use same packaging machine: $x_2 + 3x_3 \leq 600$

Objective:

$$\text{max } x_1 + 6x_2 + 13x_3$$

Constraints:

$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 + x_3 \leq 400$$
$$x_2 + 3x_3 \leq 600$$
$$x_1, x_2, x_3 \geq 0$$

Need to visualize in 3D...
How do we know that a solution is optimal?

1. Suppose your friend claims that $3100 is the optimum for

$$\max \quad x_1 + 6x_2 + 13x_3$$

and that this is achieved with $x_1 = 0, x_2 = 300, x_3 = 100$.

2. Revisit constraints to certify that solution if optimal:

$$x_1 \leq 200 \quad (1)$$
$$x_2 \leq 300 \quad (2)$$
$$x_1 + x_2 + x_3 \leq 400 \quad (3)$$
$$x_2 + 3x_3 \leq 600 \quad (4)$$

3. Note that $0 \cdot \text{Eq. (1)} + 1 \cdot \text{Eq. (2)} + 1 \cdot \text{Eq. (3)} + 4 \cdot \text{Eq. (4)}$ is

$$x_1 + 6x_2 + 13x_3 \leq 3100$$

4. But how did we come up with the coefficients $(0, 1, 1, 4)$?
Duality

- Back to simpler example: \( \text{max } x_1 + 6x_2 \) subject to

\[
\begin{align*}
  x_1 & \leq 200 \\
  x_2 & \leq 300 \\
  x_1 + x_2 & \leq 400 \\
  x_1, x_2 & \geq 0
\end{align*}
\]

- Claim that optimal solution has value 1900 where \( x_1 = 100, x_2 = 300 \)

- Adding one copy of Eq. (1) and seven copies of Eq. (2) gives

\[ x_1 + 7x_2 \leq 2300 \]

and so \( x_1 + 6x_2 \leq 2300 \) because \( x_1, x_2 \geq 0 \)

- Adding five copies of Eq. (2) and one copy of Eq. (3) gives

\[ x_1 + 6x_2 \leq 1900 \]
More Duality

1. Trying to find multipliers that give good upper bound:

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 200$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 300$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + x_2 \leq 400$</td>
</tr>
</tbody>
</table>

gives inequality $(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$.

2. If $y_1 + y_3 \geq 1$, $y_2 + y_3 \geq 6$, $y_1, y_2, y_3 \geq 0$, then an upper bound is

$$200y_1 + 300y_2 + 400y_3$$

3. Finding best such upper bound is new LP!

Minimize: $200y_1 + 300y_2 + 400y_3$

subject to

$$y_1 + y_3 \geq 1, \quad y_2 + y_3 \geq 6, \quad y_1, y_2, y_3 \geq 0$$
Duality in General

Primal and Dual Linear Programs:

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c^T x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c^T$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Theorem

Let $\text{OPT}_{\text{primal}}$ be optimal solution of Primal LP and let $\text{OPT}_{\text{dual}}$ be optimal solution of Dual LP:

$$\text{OPT}_{\text{primal}} = \text{OPT}_{\text{dual}}$$

and hence, any feasible solution of the dual LP upper bounds $\text{OPT}_{\text{primal}}$. 