CMPSCI 611: Advanced Algorithms

Lecture 21: Metric Traveling Salesperson

Andrew McGregor

Last Compiled: December 12, 2017

Outline

Metric Traveling Salesperson Problem

- Input: Weighted complete graph $G = (V, E)$ with positive weights such that for edges $e = (u, v), e' = (v, w)$, and $e'' = (u, w)$

$$w_e + w_{e'} \geq w_{e''}$$

- Goal: Find the tour (a path that visits every node exactly once and returns to starting point) of minimum total weight.

Metric TSP Approximation Algorithm

Algorithm

1. *Compute minimum spanning tree T_{mst} of G*
2. *Consider a "pseudo-tour" that walks around T_{mst}*
3. *Create tour from pseudo-tour by skipping pre-visited nodes*

Theorem

The algorithm is a 2-approximation.

Proof.

- Cost of pseudo-tour is twice cost of T_{mst}
- Cost of tour found is at most cost of pseudo-tour:

$$\text{cost}(\text{tour found}) \leq \text{cost}(\text{pseudo tour}) = 2 \cdot \text{cost}(T_{mst})$$

- Cost of T_{mst} is at most cost of optimal tour since removing an edge in an optimal tour gives a spanning tree:

$$\text{cost}(T_{mst}) \leq \text{cost}(\text{optimal tour})$$

Outline

Metric TSP 2-approx

Metric TSP 3/2 approximate

Eulerian Tours

Definition

A Eulerian tour is a path that traverses every edge of a graph exactly once and returns back to the initial vertex.

Lemma

A graph contains an Eulerian tour iff G is connected and every vertex has even degree.

Metric TSP Approximation Algorithm

Algorithm

1. *Compute minimum spanning tree* T_{mst} *of* G
2. *Let* D *be the nodes in* T_{mst} *that have odd degree*
3. *Find minimum cost perfect matching* M *on nodes of* D
4. *Find Euler tour of* $T_{mst} + M$
5. *Transform into tour by short-cutting repeated vertices.*

Analysis

Theorem

The algorithm is a 3/2-approximation and runs in polynomial time.

Proof.

- Cost of tour found is at most cost of Euler tour

$$\text{cost(tour found)} \leq \text{cost(Euler tour)} = \text{cost}(T_{mst}) + \text{cost}(M)$$

- As before, $\text{cost}(T_{mst}) \leq \text{cost(optimal tour)}$
- Cost of M is at most half cost of optimal tour

$$\text{cost}(M) \leq \text{cost(optimal tour)}/2$$

Let $D = \{d_1, \ldots, d_k\}$ be ordered according to optimal tour.

$$\begin{aligned}
\text{cost(optimal tour)} &\geq w_{d_1,d_2} + w_{d_2,d_3} + \ldots + w_{d_k,d_1} \\
&= (w_{d_1,d_2} + w_{d_3,d_4} + \ldots w_{d_{k-1},d_k}) + \\
&\qquad (w_{d_2,d_3} + w_{d_4,d_5} + \ldots w_{d_k,d_1})
\end{aligned}$$

□