Outline

Finishing NP-Completeness Reductions
Problem: Subset-Sum

- **Input:** A set $S$ of $n$ integers \( \{s_1, s_2, \ldots, s_n\} \) and a target integer $t$.
- **Question:** Is there a subset $S' \subset S$ such that $t = \sum_{s \in S'} s$?
Subset-Sum is NP-Complete

Theorem
Subset-Sum is NP-Complete

Proof.
1. It is easy to show Subset-Sum is in NP
2. It suffices to show 3-SAT $\leq_P$ Subset-Sum
3. Given $(l_{1,1} \lor l_{1,2} \lor l_{1,3}) \land (l_{2,1} \lor l_{2,2} \lor l_{2,3}) \land \ldots \land (l_{m,1} \lor l_{m,2} \lor l_{m,3})$ in $n$ variables, define the set of integers:

$$s_i = 10\ldots0y_m \ldots y_1, \quad s'_i = 10\ldots0z_m \ldots z_1$$

where $y_j = 1$ if $x_i$ if a literal in $j$-th clause and 0 otherwise and $z_j = 1$ if $\bar{x}_i$ if a literal in $j$-th clause and 0 otherwise. Let

$$t = 1\ldots13\ldots3$$

$$h_1 = 1, h_2 = 1, h_3 = 10, h_4 = 10, \ldots, h_{2m} = 10\ldots0$$
Example

For formula $\phi = (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor \overline{x_4}) \land (x_2 \lor \overline{x_3} \lor x_4)$, define

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<tbody>
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and $h_1 = 1$, $h_2 = 1$, $h_3 = 10$, $h_4 = 10$, $h_5 = 100$, $h_6 = 100$.

There exists a subset of $\{s_1, s'_1, \ldots, s_4, s'_4, h_1, h_2, h_3, h_4, h_5, h_6\}$ that sums to $t$ iff $\phi$ is satisfiable.
\( \phi \) is satisfiable implies there is a subset \( S' \) that sums to \( t \)

Suppose \( \phi \) is satisfiable:

1. Fix a satisfying assignment
2. Let \( S' = \{ s_i : x_i = \text{TRUE} \} \cup \{ s'_i : \bar{x}_i = \text{TRUE} \} \)
3. So far, for some \( 3 \geq a_i \geq 1 \):
   \[
   \sum_{s \in S'} s = (1, \ldots, 1, a_m, a_{m-1}, \ldots, a_1)
   \]
4. Can add "h" elements to \( S' \) such that
   \[
   \sum_{s \in S'} s = (1, \ldots, 1, 3, \ldots, 3) = t
   \]
Subset $S'$ that sums to $t$ implies $\phi$ is satisfiable

Suppose $\sum_{s \in S'} s = t$:

1. For each $i \in [n]$, exactly one of $s_i$ and $s'_i$ are in $S'$
2. Let $x_i$ be TRUE if $s_i \in S'$ and FALSE otherwise
3. Since there are only two "h" elements corresponding to each clause, each clause must be satisfied.
Other NP-Complete Problems

Many other problems are NP-Complete:

1. Longest path (whereas shortest path is in $P$)
2. 3D matching (whereas matching is in $P$)
3. 3-SAT (whereas 2-SAT is in $P$)
4. Max-Cut (whereas Min-Cut is in $P$)