CMPSCI 611: Advanced Algorithms

Lecture 14: Min-Cut

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Outline

Karger's Randomized Min-Cut Algorithm

Min-Cut Problem

Given an unweighted, multi-graph $G = (V, E)$, we want to partition V into V_1 and V_2 such that $|E \cap (V_1 \times V_2)|$ is minimized.

Algorithm

- *Contract a random edge $e = (u, v)$ and remove self-loops but not multi-edges*
- *Repeat until there are only 2 vertices remaining.*
- *Output the number of remaining edges.*

Let $|V| = n$ and $|E| = m$.

Correctness with low probability

Theorem

The algorithm is correct with probability at least $2/n^2$ and never an underestimate.

Proof.

- Minimum cut of the graph doesn't decrease.
- Let $C = (V_1, V_2)$ be a specific minimum cut with $|C| = k$.
- Let A_i be event that we don't contract edge across C at step i.

$$\mathbb{P}\left[\cap_{1 \le i \le n-2} A_i\right] = \mathbb{P}\left[A_1\right]\mathbb{P}\left[A_2 | A_1\right] \ldots \mathbb{P}\left[A_{n-2} | \cap_{1 \le i \le n-3} A_i\right]$$

- Number of edges before i-th step if no edges across C have been contracted so far is at least $(n-i+1)k/2$ since there are $n-i+1$ nodes remaining each with degree $\geq k$
- $\mathbb{P}\left[A_i | A_1 \cap A_2 \cap \ldots \cap A_{i-1}\right] \geq 1 - 2/(n-i+1)$ and so

$$\begin{aligned}
\mathbb{P}\left[\cap_{1 \le i \le n-2} A_i\right] &\geq (1-\frac{2}{n})(1-\frac{2}{n-1})(1-\frac{2}{n-2})\ldots(1-\frac{2}{3}) \\
&= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \ldots \cdot \frac{1}{3} = \frac{2}{n(n-1)}
\end{aligned}$$

Min-Cut Problem: Boosting the probability

Theorem

Repeating $\alpha n^2/2$ times (with new random coin flips) and returning smallest cut is correct with probability at least $1 - e^{-\alpha}$.

Proof.

- Because each repeat is independent,

$$\mathbb{P}\left[\text{always fails}\right] = \prod_{1 \le i \le \alpha n^2/2} \mathbb{P}\left[i\text{-th try fails}\right] \le (1 - 2/n^2)^{\alpha n^2/2}$$

- Use fact $1 - x \le e^{-x}$ for $x \ge 0$ and simplify.

□