Outline

Karger’s Randomized Min-Cut Algorithm
Min-Cut Problem

Given an unweighted, multi-graph \( G = (V, E) \), we want to partition \( V \) into \( V_1 \) and \( V_2 \) such that \(|E \cap (V_1 \times V_2)|\) is minimized.

Algorithm

- **Contract** a random edge \( e = (u, v) \) and remove self-loops but not multi-edges
- Repeat until there are only 2 vertices remaining.
- Output the number of remaining edges.

Let \(|V| = n\) and \(|E| = m\).
Correctness with low probability

**Theorem**

The algorithm is correct with probability at least $2/n^2$ and never an underestimate.

**Proof.**

- Minimum cut of the graph doesn't decrease.
- Let $C = (V_1, V_2)$ be a specific minimum cut with $|C| = k$.
- Let $A_i$ be event that we don't contract edge across $C$ at step $i$.

$$
\mathbb{P} \left[ \cap_{1 \leq i \leq n-2} A_i \right] = \mathbb{P} \left[ A_1 \right] \mathbb{P} \left[ A_2 | A_1 \right] \ldots \mathbb{P} \left[ A_{n-2} | \cap_{1 \leq i \leq n-3} A_i \right]
$$

- Number of edges before $i$-th step if no edges across $C$ have been contracted so far is at least $(n - i + 1)k/2$
- $\mathbb{P} \left[ A_i | A_1 \cap A_2 \cap \ldots \cap A_{i-1} \right] \geq 1 - 2/(n - i + 1)$ and so

$$
\mathbb{P} \left[ \cap_{1 \leq i \leq n-2} A_i \right] \geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right)\ldots\left(1 - \frac{2}{3}\right)
$$

$$
= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \ldots \cdot \frac{1}{3} = \frac{2}{n(n-1)}
$$
Min-Cut Problem: Boosting the probability

Theorem
Repeating $\alpha n^2/2$ times (with new random coin flips) and returning smallest cut is correct with probability at least $1 - e^{-\alpha}$.

Proof.

- Because each repeat is independent,

  $$\mathbb{P} [\text{always fails}] = \prod_{1 \leq i \leq \alpha n^2/2} \mathbb{P} [\text{i-th try fails}] \leq (1 - 2/n^2)^{\alpha n^2/2}$$

- Use fact $1 - x \leq e^{-x}$ for $x \geq 0$ and simplify.