Definitions

Input:
- Directed Graph $G = (V, E)$
- Capacities $C(u, v) > 0$ for $(u, v) \in E$ and $C(u, v) = 0$ for $(u, v) \notin E$
- A source node $s$, and sink node $t$
Capacity

\[
\begin{align*}
v_1 &\quad v_2 \\
v_3 &\quad v_4 \\
16 &\quad 12 \\
10 &\quad 4 \\
13 &\quad 9 \\
14 &\quad 7 \\
20 &\quad 4 \\
\end{align*}
\]
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Output: A flow $f$ from $s$ to $t$ where $f : V \times V \rightarrow \mathbb{R}$ satisfies
- Skew-symmetry: $\forall u, v \in V, f(u, v) = -f(v, u)$
- Conservation of Flow: $\forall v \in V - \{s, t\}, \sum_{u \in V} f(u, v) = 0$
- Capacity Constraints: $\forall u, v \in V, f(u, v) \leq C(u, v)$

Goal: Maximize “size of the flow”, i.e., the total flow coming leaving $s$:

$$|f| = \sum_{v \in V} f(s, v)$$
Capacity
Definition
An $s-t$ cut of $G$ is a partition of the vertices into two sets $A$ and $B$ such that $s \in A$ and $t \in B$.

Definition
The capacity of a cut $(A, B)$ is

$$C(A, B) = \sum_{u \in A, v \in B} C(u, v)$$

Definition
The flow across a cut $(A, B)$ is

$$f(A, B) = \sum_{u \in A, v \in B} f(u, v)$$

Note that because of capacity constraints: $f(A, B) \leq C(A, B)$
First Cut
Second Cut
Max-Flow Min-Cut

Lemma
For any flow $f$: for all $s$-$t$ cuts $(A, B)$, $f(A, B)$ equals $|f|$.

Theorem (Max-Flow Min-Cut)
For any flow network and flow $f$, the following statements are equivalent:
1. $f$ is a maximum flow.
2. There exists an $s$-$t$ cut $(A, B)$ such that $|f| = C(A, B)$

We’ll prove both next class.
Residual Networks and Augmenting Paths

Residual network encodes how you can change the flow between two nodes given the current flow and the capacity constraints.

Definition
Given a flow network \( G = (V, E) \) and flow \( f \) in \( G \), the residual network \( G_f \) is defined as

\[
G_f = (V, E_f) \text{ where } E_f = \{(u, v) : C(u, v) - f(u, v) > 0\}
\]

\[
C_f(u, v) = C(u, v) - f(u, v)
\]

Note that \((u, v) \in E_f\) implies either \(C(u, v) > 0\) or \(C(v, u) > 0\).

Definition
An augmenting path for flow \( f \) is a path from \( s \) to \( t \) in graph \( G_f \). The bottleneck capacity \( b(p) \) is the minimum capacity in \( G_f \) of any edge of \( p \). We can increase flow by \( b(p) \) along an augmenting path.
Capacity/Flow

16/11
13/8
14/11
9/4
12/12
4/10
7/7
20/15
4/4
14/11
Augmenting Path
Old Flow

\begin{itemize}
  \item $s$ to $v_1$: $16/11$
  \item $s$ to $v_3$: $13/8$
  \item $v_1$ to $v_2$: $12/12$
  \item $v_1$ to $v_3$: $10/0$
  \item $v_1$ to $v_4$: $4/1$
  \item $v_3$ to $v_4$: $14/11$
  \item $v_3$ to $t$: $9/4$
  \item $v_2$ to $t$: $20/15$
  \item $v_2$ to $v_4$: $7/7$
  \item $t$ to $v_4$: $4/4$
\end{itemize}
New Flow

Diagram of a flow network with vertices $s$, $v_1$, $v_2$, $v_3$, and $v_4$. The edges are labeled with capacities:
- From $s$ to $v_1$: 16/11
- From $s$ to $v_3$: 13/12
- From $v_1$ to $v_2$: 12/12
- From $v_3$ to $v_4$: 14/11
- From $v_3$ to $v_1$: 10/0
- From $v_1$ to $v_3$: 4/1
- From $v_2$ to $v_4$: 20/19
- From $v_2$ to $t$: 7/7
- From $v_4$ to $t$: 4/4

The source vertex is $s$, and the sink vertex is $t$. The capacities of the edges indicate the maximum flow that can pass through each edge.
Min Capacity Cut Proves this is Optimal
Old Residual Graph

\[ v_1 \quad v_2 \]
\[ v_3 \quad v_4 \]
\[ s \quad t \]
New Residual Graph
Ford-Fulkerson Algorithm

Algorithm

1. \( \text{flow } f = 0 \)
2. while there exists an augmenting path \( p \) for \( f \)
   2.1 find augmenting path \( p \)
   2.2 augment \( f \) by \( b(p) \) units along \( p \)
3. return \( f \)

Theorem

The algorithm finds a maximum flow in time \( O(|E||f^*|) \) if capacities are integral where \( |f^*| \) is the size of the maximum flow.

Proof.

\( O(|E|) \) time to find each augmenting path via BFS and \( |f^*| \) iterations because each augmenting path increases flow by at least 1.