Definitions

Input:
- Directed Graph $G = (V, E)$
- Capacities $C(u, v) > 0$ for $(u, v) \in E$ and $C(u, v) = 0$ for $(u, v) \notin E$
- A source node $s$, and sink node $t$
Capacity

Diagram with nodes labeled as $s$, $v_1$, $v_2$, $v_3$, $v_4$, and $t$. Edges with capacities labeled as: $s$ to $v_1$: 16, $v_1$ to $v_2$: 12, $v_1$ to $v_3$: 10, $v_3$ to $v_4$: 14, $v_3$ to $s$: 13, $v_2$ to $t$: 20, $v_2$ to $v_4$: 7, $v_4$ to $t$: 4.
Definitions

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- Capacities $C(u, v) > 0$ for $(u, v) \in E$ and $C(u, v) = 0$ for $(u, v) \not\in E$
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Output: A flow $f$ from $s$ to $t$ where $f : V \times V \to \mathbb{R}$ satisfies
- Skew-symmetry: $\forall u, v \in V, f(u, v) = -f(v, u)$
- Conservation of Flow: $\forall v \in V \setminus \{s, t\}, \sum_{u \in V} f(u, v) = 0$
- Capacity Constraints: $\forall u, v \in V, f(u, v) \leq C(u, v)$

Goal: Maximize “size of the flow”, i.e., the total flow coming leaving $s$:

$$|f| = \sum_{v \in V} f(s, v)$$
Capacity
Capacity/Flow

Graph with nodes labeled \( s \), \( v_1 \), \( v_2 \), \( v_3 \), and \( v_4 \), and edges with capacities 16/11, 13/8, 10/0, 4/1, 14/11, 9/4, 7/7, 20/15, 4/4, and 12/12.
Cut Definitions

Definition
An $s - t$ cut of $G$ is a partition of the vertices into two sets $A$ and $B$ such that $s \in A$ and $t \in B$.

Definition
The capacity of a cut $(A, B)$ is

$$C(A, B) = \sum_{u \in A, v \in B} C(u, v)$$

Definition
The flow across a cut $(A, B)$ is

$$f(A, B) = \sum_{u \in A, v \in B} f(u, v)$$

Note that because of capacity constraints: $f(A, B) \leq C(A, B)$
First Cut

Diagram with nodes labeled as follows:

- Node 1: v1
- Node 2: v2
- Node 3: v3
- Node 4: v4
- Source node: s
- Target node: t

Edge labels:

- Source to v1: 16/11
- v1 to v2: 12/12
- v2 to t: 20/15
- s to v3: 10/0
- v3 to v1: 4/1
- v3 to v4: 10/0
- v4 to t: 4/4
- v4 to v3: 14/11
- v2 to v4: 7/7

Additional notes:

- v1 and v2 are connected directly.
- s and v3 are connected directly.
- v3 and v4 are connected directly.
- v4 and t are connected directly.
Second Cut
Max-Flow Min-Cut

Lemma
For any flow $f$: for all $s$-$t$ cuts $(A, B)$, $f(A, B)$ equals $|f|$.

Theorem (Max-Flow Min-Cut)
For any flow network and flow $f$, the following statements are equivalent:

1. $f$ is a maximum flow.
2. There exists an $s$ – $t$ cut $(A, B)$ such that $|f| = C(A, B)$

We’ll prove both next class.
Residual Networks and Augmenting Paths

Residual network encodes how you can change the flow between two nodes given the current flow and the capacity constraints.

Definition

Given a flow network $G = (V, E)$ and flow $f$ in $G$, the residual network $G_f$ is defined as

$$G_f = (V, E_f) \text{ where } E_f = \{(u, v) : C(u, v) - f(u, v) > 0\}$$

$$C_f(u, v) = C(u, v) - f(u, v)$$

Note that $(u, v) \in E_f$ implies either $C(u, v) > 0$ or $C(v, u) > 0$.

Definition

An augmenting path for flow $f$ is a path from $s$ to $t$ in graph $G_f$. The bottleneck capacity $b(p)$ is the minimum capacity in $G_f$ of any edge of $p$. We can increase flow by $b(p)$ along an augmenting path.
Capacity/Flow

The diagram represents a network flow problem with capacities and flows on the edges. The nodes are labeled as follows: s, v₁, v₂, v₃, and t. The capacities and flows are indicated on the edges.

- From s to v₁: 16/11
- From v₁ to v₂: 12/12
- From v₂ to t: 20/15
- From s to v₃: 13/8
- From v₃ to s: 10/0
- From v₃ to v₁: 4/1
- From v₃ to v₄: 14/11
- From v₁ to v₃: 9/4
- From v₂ to v₄: 7/7
- From v₄ to v₂: 4/4

The network shows the flow and capacity constraints between the nodes.
Residual
Augmenting Path

\[ \text{Augmenting Path} \]

\[ \begin{align*}
  s & \rightarrow v_3 \rightarrow v_2 \rightarrow t \\
  s & \rightarrow v_1 \rightarrow v_2 \rightarrow t \\
  s & \rightarrow v_3 \rightarrow v_4 \rightarrow t \\
  s & \rightarrow v_1 \rightarrow v_4 \rightarrow t \\
  s & \rightarrow v_1 \rightarrow v_3 \rightarrow \text{Augmenting Path}\n\end{align*} \]
Old Flow
New Flow

\begin{figure}
\centering
\includegraphics[width=\textwidth]{network}
\caption{Diagram of the network flow.}
\end{figure}
Min Capacity Cut Proves this is Optimal
Old Residual Graph
New Residual Graph
Ford-Fulkerson Algorithm

Algorithm

1. flow \( f = 0 \)
2. while there exists an augmenting path \( p \) for \( f \)
   2.1 find augmenting path \( p \)
   2.2 augment \( f \) by \( b(p) \) units along \( p \)
3. return \( f \)

Theorem
The algorithms finds a maximum flow in time \( O(|E||f^*|) \) if capacities are integral where \( |f^*| \) is the size of the maximum flow.

Proof.
\( O(|E|) \) time to find each augmenting path via BFS and \( |f^*| \) iterations because each augmenting path increases flow by at least 1. □