Outline

Summary of Matroid Results
The Problem

Definition
A subset system $S = (E, \mathcal{I})$ is a finite set $E$ with a collection $\mathcal{I}$ of subsets of $E$ such that if $i \in \mathcal{I}$ and $i' \subset i$ then $i' \in \mathcal{I}$.

Problem Given a subset system $S = (E, \mathcal{I})$ and weight function $w : E \rightarrow \mathbb{R}^+$, find $i \in \mathcal{I}$ such that $w(i) = \sum_{e \in i} w(e)$ is maximized.

Algorithm (Greedy)
1. $i = \emptyset$
2. Sort elements of $E$ by non-increasing weight
3. For each $e \in E$: If $i + e \in \mathcal{I}$ then $i \leftarrow i + e$
Matroid Definition and Theorem

**Definition**
Subset system $(E, \mathcal{I})$ has the **exchange property** if

$$\forall i, i' \in \mathcal{I} : (|i| < |i'|) \implies (\exists e \in i' - i \text{ such that } i + e \in \mathcal{I})$$

**Definition**
A subset system $(E, \mathcal{I})$ has the **cardinality property** if

$$\forall A \subseteq E : (i, i' \in \mathcal{I} \text{ maximal subsets of } A) \implies (|i| = |i'|)$$

where we say $i \in \mathcal{I}$ is a **maximal subset of** $A$ if $i \subseteq A$ and there doesn’t exist $e \in A$ such that $i + e \in \mathcal{I}$.

**Theorem**
*Given a subset system $(E, \mathcal{I})$, the following statements are equivalent:*

1. **Greedy algorithm returns optimal solution for any weight function.**
2. **The subset system obeys the exchange property.**
3. **The subset system obeys the cardinality property.**
Exchange Property implies Cardinality Property

- Let $i, i'$ be maximal subsets of $A \subseteq E$. Need to show $|i| = |i'|$
- If $|i'| > |i|$, the exchange property implies

  $$\exists e \in i' - i \text{ such that } i + e \in \mathcal{I}$$

- Thus $i$ was not maximal in $A$. Contradiction!
Cardinality Property implies Exchange Property

- Suffices to show that \((E, \mathcal{I})\) not a matroid implies there exists \(A\) and \(i, j \in \mathcal{I}\) such that \(|i| \neq |j|\) and \(i, j\) are maximal in \(A\)
- \((E, \mathcal{I})\) not a matroid implies that
  
  \[ \exists \ i, i' \in \mathcal{I} \text{ such that } |i| < |i'| \text{ and } \forall e \in i' - i \text{ with } i + e \in \mathcal{I} \]

- Define \(A = i \cup i'\) and note that \(i\) is maximal in \(A\).
- There exists \(j \in \mathcal{I}\) such that \(i' \subseteq j\) and \(j\) is maximal in \(A\).
- But \(|j| \geq |i| + 1\) as required.
Example 1

Theorem

*The Maximum Weight Forest (MWF) subset system is a matroid.*

Proof.

- Pick an arbitrary subset of edges $A \subseteq E$.
- Let $n_1, \ldots, n_k$ be the size of the connected components.
- Any maximal acyclic subset of $A$ has size

\[(n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) = n - k\]

because a maximal acyclic subgraph of a connected graph on $n_i$ nodes is a tree and has $n_i - 1$ edges.

- Cardinality Theorem implies that it’s a matroid.

\[\square\]
Example 2

**Theorem**

Let $E$ be a set of directed edges and $\mathcal{I}$ be subsets such that no two edges in the same subset point to the same node. This is a matroid.

**Proof.**

- For any $A \subseteq E$, the number of edges in a maximal subset of $A$ is equal to the number of vertices pointed to in $A$.
- Cardinality Theorem implies that it’s a matroid.