Greedy Algorithms Overview

“An algorithm that finds a solution by adding elements one by one, where each element that is added is the best current choice without regard to the future consequences of this choice.”

- Minimum Spanning Tree and Kruskal’s algorithm
- Matroids and Subset Systems
- Bipartite Matching and Intersections of Matroids
- Union-Find Data Structure
Minimum Spanning Tree and Kruskal’s Algorithm

Problem: Given an undirected, connected graph \((V, E)\) with edge weights find the min-weight subset \(E' \subset E\) such that the graph \((V, E')\) is acyclic and connected, i.e., a min-weight spanning tree (MST).

Throughout this class we’ll assume all edge weights are distinct although everything generalizes to when some weights are the same.

Algorithm (Kruskal)

1. *Sort edges by increasing weight*
2. \(F = \emptyset\)
3. Until \(F\) is a spanning tree of \(G\)
   3.1 Get the next edge \(e\)
   3.2 If \(F + e\) is acyclic then \(F = F + e\)

The algorithm produces a tree because a) it never completes a cycle so the end result is acyclic and b) for any edge \((u, v)\) in a tree of the original graph, either \((u, v)\) is added or there is a path in \(F\) between \(u\) and \(v\).
Running Time of Kruskal’s Algorithm

Implementation: Maintain an array $A$ with an entry for each $v \in V$ that indicates which connected component it belongs to.

- **Sorting:** $O(|E| \log |E|)$
- **Checking if acyclic:** $|E|$ checks and each is $O(1)$ time.
- **Adding $e$ to $F$:** Updating array takes $O(|V|)$ time and array is updated exactly $|V| - 1$ times.

**Total Running Time:** $O(|E| \log |E| + |V|^2)$

Will make this $O(|E| \log |E|)$ later via the union-find data structure.
Proof of Correctness: Part 1

**Cut Lemma:** Let $S \subset V$ and let $e = (u, v)$ be the lightest edge such that $u \in S$ and $v \notin S$. The MST contains edge $e$.

**Proof:**

- Suppose there exists a minimum spanning tree $T$ that doesn’t include $e$. We’ll construct a different spanning tree $T'$ such that $w(T') < w(T)$ and hence $T$ can’t be the MST.

- Since $T$ is a spanning tree, there’s a $u \leadsto v$ path $P$ in $T$. Since the path starts in $S$ and ends up outside $S$, there must be an edge $e' = (u', v')$ on this path where $u' \in S$, $v' \notin S$.

- Let $T' = T - \{e'\} + \{e\}$. This is still spanning tree, since any path in $T$ that needed $e'$ can be routed via $e$ instead. But since $e$ was the lightest edge between $S$ and $V \setminus S$,

  $$w(T') = w(T) - w(e') + w(e) < w(T) - w(e') + w(e') = w(T)$$
Proof of Correctness: Part 2

**Kruskal’s Algorithm:** Sort the edges by increasing weight and keep on adding the next edge that doesn’t complete a cycle.

**Proof of Correctness:**
- Suppose $e = (u, v)$ is the next edge added.
- Let $S$ be the set of nodes that can be reached from $u$ before $e$ was added. Note that $v \not \in S$ since otherwise adding $e$ would have completed a cycle.
- No other edge between $S$ and $V \setminus S$ can have been encountered before since if it had it would have been added since it doesn’t complete a cycle. Hence $e$ is the lightest edge between $S$ and $V \setminus S$. Therefore, the cut lemma implies $e$ must be in the MST.
A Different Greedy Algorithm: Prim’s Algorithm

Prim’s Algorithm:
- Sort the edges by increasing weight.
- Let $S = \{s\}$.
- While $S \neq V$: Add next edge $(u, v)$ where $u \in S$, $v \notin S$ and add $v$ to $S$.

Proof of Correctness:
- Let $S$ be the set of nodes in the tree constructed so far.
- The next edge added to the tree is the lightest edge between $S$ and $V \setminus S$. Hence, the cut lemma implies $e$ must be in the MST.