NAME: ____________________________

# CMPSCI 611
# Advanced Algorithms
# Final Exam Fall 2010

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DIRECTIONS:

- Do not turn over the page until you are told to do so.
- This is a *closed book exam.* No communicating with other students, or looking at notes, or using electronic devices. You may ask the professor to clarify the meaning of a question but do so in a way that causes minimal disruption.
- If you finish early, you may leave early but do so as quietly as possible. The exam script should be given to the professor.
- There are five questions. All carry the same number of points but some questions may be easier than others. Don't spend too long on a problem if you're stuck – you may find that there are other easier questions.
- The front and back of the pages can be used for solutions. There are also a couple of blank pages at the end that can be used. If you are using these pages, clearly indicate which question you're answering. Further paper can be requested if required.
- The exam will finish at 3:30 pm.
- Good luck!

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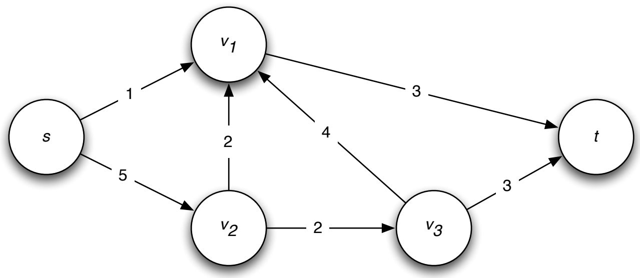
|---|---|
| 1 | /10 |
| 2 | /10 |
| 3 | /10 |
| 4 | /10 |
| 5 | /10 |
| Total | /50 |

**Question 1:** In the first part of this question, consider the following flow network where the value of each edge represents the capacity of that edge:




1. What is the value of maximum flow possible from $s$ to $t$? No proof required.

2. What is the value of the minimum $s$-$t$ cut? No proof required.

The next part of this question concerns an arbitrary flow network. For each of the following statements, write whether they are true or false.

3. If the capacity of each edge in the network is doubled, the maximum flow is doubled.

4. If the capacity of each edge that leaves the source $s$ is increased, the maximum flow is increased.

**Question 2:** In this question, the goal to provide examples where the suggested algorithm **does not** work. *You should specify both the correct answer and the result of the algorithm.*

1. Given a graph $G = (V, E)$ where each node $v$ has a weight $w(v)$, we want to find the maximum weight independent set. Show a counter-example to the greedy algorithm. **Hint:** Recall that a maximum weight independent set is a subset of nodes $V' \subseteq V$ such that $\sum_{v \in V'} w(v)$ is maximized subject to the condition that $(u, v) \notin E$ for any $u, v \in V'$.

2. Given a 3-CNF formula $\phi$ with variables $x_1, \ldots, x_n$ and an odd number of clauses, we want to determine whether $\phi$ is satisfiable. Consider the algorithm that assigns each $x_i$ to TRUE if there are at least as many occurrences of the literal $x_i$ as the literal $\bar{x}_i$. Show a satisfiable formula for which the algorithm does not find a satisfying assignment.

3. Dijkstra's algorithm does not necessarily work on graphs with negative weights. Give an example of a graph with positive and negative weights (but no negative weight cycles) such that Dijkstra's algorithm doesn't work. Remember to specify the source node.

**Question 3:** A pizza restaurant chain is considering opening a series of restaurants along a highway. The $n$ possible locations are along a straight line, and the distances of these locations from the start of the highway are, in miles and in increasing order, $m_1, m_2, \ldots, m_n$. The constraints are as follows:

- At each location, you may open at most one restaurant. A restaurant at location $i$ makes profit $p_i$, where $p_i > 0$ and $i = 1, 2, \ldots, n$.
- Any two restaurants should be at least $k$ miles apart, where $k$ is a positive integer.

1. Give an efficient dynamic programming algorithm to find the placement of the restaurants such that the maximum profit is achieved subject to the given constraints.

2. Suppose $p_1 > p_2 > \ldots > p_n$ and $m_1 = 0, m_2 = 1, \ldots, m_n = n - 1$. What is the value of the optimal solution?

**Question 4:** Given a collection of $m$ sets $C = \{A_1, \ldots, A_m\}$ and an integer $d$, the SET-COVER problem is to determine if there are $d$ sets $A_{i_1}, \ldots, A_{i_d}$ such that $A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_d} = \cup_{A \in C} A$.

1. Prove that SET-COVER is NP-complete. You may assume VERTEX-COVER is NP-complete.

2. Consider a collection of $m$ sets $C = \{A_1, \ldots, A_m\}$ where every set is of size 3. Design a *simple* algorithm with approximation ratio at most 3 for the problem of finding the minimum $d$ such that there exist $d$ sets $A_{i_1}, \ldots, A_{i_d}$ such that $A_{i_1} \cup \ldots \cup A_{i_d} = \cup_{A \in C} A$.

**Question 5:** Consider the following randomized algorithm for MAX-CUT: Given an undirected, unweighted graph $G = (V, E)$, define a cut $(A, V \setminus A)$ by independently putting each node $v \in V$ in $A$ with probability $1/2$. Let $X$ be the number of edges that cross the cut, i.e.,

$$X = |\{(u, v) \in E : u \in A, v \notin A\}| \ .$$

1. State Chebyshev's inequality.

2. What is the expected value of $X$? Show your working.

3. What is the variance of $X$? Hint: Recall that if random variables $Z_1, \ldots, Z_m$ are such that $Z_i$ and $Z_j$ are independent for any $i \neq j$, then $\mathbb{V}(Z_1 + \ldots + Z_m) = \sum_{i=1}^{m} \mathbb{V}(Z_i)$. Remember to show that the relevant random variables are independent.

4. Prove a *lower bound* on $\mathbb{P}\left[|X - m/2| < t\right]$ where $m$ is the number of edges in $G$.