COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor
Lecture 7
DISTINCT ELEMENTS RECAP

- Estimate \( \# \) distinct elements \( d \) in stream \( x_1, \ldots, x_n \in U \)
• Estimate the number of distinct elements $d$ in stream $x_1, \ldots, x_n \in U$

• **Basic Algorithm:**
  - Let $h_1, h_2, \ldots, h_k : U \rightarrow [0, 1]$ be random hash functions.
  - Compute $s = \frac{1}{k} \sum s_i$ where $s_i = \min_{j \in [n]} h_i(x_j)$
  - Return $\hat{d} = 1/s - 1$
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• Median Trick: If an algorithm returns a sufficiently accurate numerical answer with probability at least \(3/4\), run it \(O(\log(1/\delta))\) times and take the median answer. This will have the required accuracy with probability at least \(1 - \delta\).
Questions on distinct elements counting?
**Jaccard Index:** A similarity measure between two sets.

\[ J(A, B) = \frac{|A \cap B|}{|A \cup B|} = \frac{\text{# shared elements}}{\text{# total elements}}. \]

Natural measure for similarity between bit strings – interpret an \( n \) bit string as a set, containing the elements corresponding the positions of its ones. \( J(x, y) = \frac{\text{# shared ones}}{\text{total ones}} \).
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- **All-pairs Similarity Search**: Have \( n \) different sets/bit strings and want to find all pairs with high Jaccard similarity. \( \Omega(n^2) \) time if we check all pairs explicitly.

Will speed up via randomized locality sensitive hashing.
Document Similarity:

- E.g., detecting plagiarism, copyright infringement, spam.
- Use Shingling + Jaccard similarity.

The quick brown fox jumped high. 

\[ \{ \text{the quick brown fox} \} \]
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- **Twitter**: represent a user as the set of accounts they follow. Match similar users based on the Jaccard similarity of these sets. Recommend that you follow accounts followed by similar users. **Netflix**: look at sets of movies watched. **Amazon**: look at products purchased, etc.
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See Section 3.8.2 of Mining Massive Datasets for a discussion of a real world example involving 1 million customers. Naively this would be $\binom{1000000}{2} \approx 500$ billion pairs of customers to check!
Many applications to spam/fraud detection. E.g.

- **Fake Reviews**: Very common on websites like Amazon. Detection often looks for (near) duplicate reviews on similar products, which have been copied. 'Near duplicate' measured with shingles + Jaccard similarity.

- **Lateral phishing**: Phishing emails sent to addresses at a business coming from a legitimate email address at the same business that has been compromised.

  One method of detection looks at the recipient list of an email and checks if it has small Jaccard similarity with any previous recipient lists. If not, the email is flagged as possible spam.
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**MinHash(A):** [Andrei Broder, 1997 at Altavista]

- Let $h : U \rightarrow [0, 1]$ be a random hash function
- $s := 1$
- For $x_1, \ldots, x_{|A|} \in A$
  - $s := \min(s, h(x_k))$
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Identical to our distinct elements sketch!
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$$= \sum_{x \in A \cap B} \frac{1}{|A \cup B|} = \frac{|A \cap B|}{|A \cup B|} = J(A, B)$$
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How does locality sensitive hashing help for similarity search?

- **Near Neighbor Search**: Given an item \( x \), compute \( h(x) \). Only search for similar items in the \( h(x) \) bucket of the hash table.
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- Create a hash table of size $m$, choose a random hash function $g : [0, 1] \rightarrow [m]$, and insert each item $x$ into bucket $g(MH(x))$. Search for items similar to $y$ in bucket $g(MH(y))$. 

**What is $Pr[g(MH(z)) = g(MH(y))]$ assuming $J(z, y) \leq 1/3$ and $g$ is collision free?**

At most $1/3$

**For each document $x$ in your database with $J(x, y) \geq 1/2$ what is the probability you will find $x$ in bucket $g(MH(y))$?**

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REDUCING FALSE NEGATIVES

With a simple use of MinHash, we miss a match $x$ with $J(x, y) = 1/2$ with probability $1/2$. How can we reduce this false negative rate?

Repetition: Run MinHash $t$ times independently, to produce hash values $MH_1(x), \ldots, MH_t(x)$. Apply random hash function $g$ to map all these values to locations in $t$ hash tables.
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  \[ 1 - \text{(probability in no buckets)} = 1 - \left( \frac{3}{4} \right)^t \]
REDUCING FALSE NEGATIVES

With a simple use of MinHash, we miss a match \( x \) with \( J(x, y) = 1/2 \) with probability 1/2. How can we reduce this false negative rate?

Repetition: Run MinHash \( t \) times independently, to produce hash values \( MH_1(x), \ldots, MH_t(x) \). Apply random hash function \( g \) to map all these values to locations in \( t \) hash tables.

- To search for items similar to \( y \), look at all items in bucket \( g(MH_1(y)) \) of the 1\(^{st} \) table, bucket \( g(MH_2(y)) \) of the 2\(^{nd} \) table, etc.

- What is the probability that \( x \) with \( J(x, y) = 1/2 \) is in at least one of these buckets, assuming for simplicity \( g \) has no collisions?
  \[
  1 - \left( \text{probability in no buckets} \right) = 1 - \left( \frac{1}{2} \right)^t \approx .99 \text{ for } t = 7.
  \]

- What is the probability that \( x \) with \( J(x, y) = 1/4 \) is in at least one of these buckets, assuming for simplicity \( g \) has no collisions?
  \[
  1 - \left( \text{probability in no buckets} \right) = 1 - \left( \frac{3}{4} \right)^t \approx .87 \text{ for } t = 7.
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With a simple use of MinHash, we miss a match \( x \) with \( J(x, y) = 1/2 \) with probability \( 1/2 \). How can we reduce this false negative rate?

Repetition: Run MinHash \( t \) times independently, to produce hash values \( MH_1(x), \ldots, MH_t(x) \). Apply random hash function \( g \) to map all these values to locations in \( t \) hash tables.

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Potential for a lot of false positives! Slows down search time.
We want to balance a small probability of false negatives (a high hit rate) with a small probability of false positives (a small query time.)
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Create $t$ hash tables. Each is indexed into not with a single MinHash value, but with $r$ values, appended together. A length $r$ signature.
Consider searching for matches in $t$ hash tables, using MinHash signatures of length $r$. For $x$ and $y$ with Jaccard similarity $J(x, y) = s$: 

1. Probability that a single hash matches. 
   \[ \Pr[MH_{i, j}(x) = MH_{i, j}(y)] = J(x, y) = s. \]
2. Probability that $x$ and $y$ having matching signatures in repetition $i$. 
   \[ \Pr[MH_{i, 1}(x), \ldots, MH_{i, r}(x) = MH_{i, 1}(y), \ldots, MH_{i, r}(y)] = s^r. \]
3. Probability that $x$ and $y$ don't match in repetition $i$: 
   \[ 1 - s^r. \]
4. Probability that $x$ and $y$ don't match in all repetitions: 
   \[ (1 - s^r)^t. \]
5. Probability that $x$ and $y$ match in at least one repetition: 
   \[ \text{Hit Probability: } 1 - (1 - s^r)^t. \]
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- Probability that $x$ and $y$ having matching signatures in repetition $i$.
  $\Pr[MH_{i,1}(x), \ldots, MH_{i,r}(x) = MH_{i,1}(y), \ldots, MH_{i,r}(y)] = s^r$.

- Probability that $x$ and $y$ don’t match in repetition $i$: $1 - s^r$. 
Balancing Hit Rate and Query Time

Consider searching for matches in \( t \) hash tables, using MinHash signatures of length \( r \). For \( x \) and \( y \) with Jaccard similarity \( J(x, y) = s \):

- Probability that a single hash matches.
  \[ \Pr [MH_{i,j}(x) = MH_{i,j}(y)] = J(x, y) = s. \]

- Probability that \( x \) and \( y \) having matching signatures in repetition \( i \).
  \[ \Pr [MH_{i,1}(x), \ldots, MH_{i,r}(x) = MH_{i,1}(y), \ldots, MH_{i,r}(y)] = s^r. \]

- Probability that \( x \) and \( y \) don’t match in repetition \( i \): \( 1 - s^r \).

- Probability that \( x \) and \( y \) don’t match in all repetitions:
Consider searching for matches in \( t \) hash tables, using MinHash signatures of length \( r \). For \( x \) and \( y \) with Jaccard similarity \( J(x, y) = s \):

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- Probability that \( x \) and \( y \) having matching signatures in repetition \( i \).
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  \Pr [MH_{i,1}(x), \ldots, MH_{i,r}(x) = MH_{i,1}(y), \ldots, MH_{i,r}(y)] = s^r.
  \]

- Probability that \( x \) and \( y \) don’t match in repetition \( i \): \( 1 - s^r \).

- Probability that \( x \) and \( y \) don’t match in all repetitions: \( (1 - s^r)^t \).
Consider searching for matches in $t$ hash tables, using MinHash signatures of length $r$. For $x$ and $y$ with Jaccard similarity $J(x, y) = s$:

- Probability that a single hash matches.
  $$\Pr [MH_{i,j}(x) = MH_{i,j}(y)] = J(x, y) = s.$$  

- Probability that $x$ and $y$ having matching signatures in repetition $i$.
  $$\Pr [MH_{i,1}(x), \ldots, MH_{i,r}(x) = MH_{i,1}(y), \ldots, MH_{i,r}(y)] = s^r.$$  

- Probability that $x$ and $y$ don’t match in repetition $i$: $1 - s^r$.

- Probability that $x$ and $y$ don’t match in all repetitions: $(1 - s^r)^t$.

- Probability that $x$ and $y$ match in at least one repetition:
Consider searching for matches in $t$ hash tables, using MinHash signatures of length $r$. For $x$ and $y$ with Jaccard similarity $J(x, y) = s$:

- Probability that a single hash matches.
  \[ \Pr [MH_{i,j}(x) = MH_{i,j}(y)] = J(x, y) = s. \]

- Probability that $x$ and $y$ having matching signatures in repetition $i$.
  \[ \Pr [MH_{i,1}(x), \ldots, MH_{i,r}(x) = MH_{i,1}(y), \ldots, MH_{i,r}(y)] = s^r. \]

- Probability that $x$ and $y$ don’t match in repetition $i$: $1 - s^r$.

- Probability that $x$ and $y$ don’t match in all repetitions: $(1 - s^r)^t$.

- Probability that $x$ and $y$ match in at least one repetition:

  Hit Probability: $1 - (1 - s^r)^t$. 

Using $t$ repetitions each with a signature of $r$ MinHash values, the probability that $x$ and $y$ with Jaccard similarity $J(x, y) = s$ match in at least one repetition is: $1 - (1 - s^r)^t$. 

$\text{THE } s\text{-CURVE}$
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\[ r = 10, \quad t = 10 \]
Using $t$ repetitions each with a signature of $r$ MinHash values, the probability that $x$ and $y$ with Jaccard similarity $J(x, y) = s$ match in at least one repetition is: $1 - (1 - s^r)^t$. 

\[
 r = 5, \ t = 30
\]
Using $t$ repetitions each with a signature of $r$ MinHash values, the probability that $x$ and $y$ with Jaccard similarity $J(x, y) = s$ match in at least one repetition is: $1 - (1 - s^r)^t$.

$r$ and $t$ are tuned depending on application. ‘Threshold’ when hit probability is $1/2$ is $\approx (1/t)^{1/r}$. E.g., $\approx (1/30)^{1/5} = .51$ in this case.
For example: Consider a database with 10,000,000 audio clips. You are given a clip $x$ and want to find any $y$ in the database with $J(x, y) \geq .9$. 

- There are 10 true matches in the database with $J(x, y) \geq .9$.
- There are 10,000 near matches with $J(x, y) \in [.7, .9]$.

With signature length $r = 25$ and repetitions $t = 50$, hit probability for $J(x, y) = s$ is $1 - (1 - s)^{25/50}$.

- Hit probability for $J(x, y) \geq .9$ is $\geq 1 - (1 - .9)^{25/50} \approx .98$.
- Hit probability for $J(x, y) \in [.7, .9]$ is $\leq 1 - (1 - .7)^{25/50} \approx .98$.
- Hit probability for $J(x, y) \leq .7$ is $\leq 1 - (1 - .7)^{25/50} \approx .007$.

Expected Number of Items Scanned: (proportional to query time) $\leq 10 + .98 \times 10,000 + .007 \times 9,990 \approx 80,000 \ll 100,000$. 

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**For example:** Consider a database with 10,000,000 audio clips. You are given a clip $x$ and want to find any $y$ in the database with $J(x, y) \geq .9$.

- There are 10 **true matches** in the database with $J(x, y) \geq .9$.
- There are 10,000 **near matches** with $J(x, y) \in [.7, .9]$. 
**S-CURVE EXAMPLE**

**For example:** Consider a database with 10,000,000 audio clips. You are given a clip $x$ and want to find any $y$ in the database with $J(x, y) \geq .9$.

- There are 10 true matches in the database with $J(x, y) \geq .9$.
- There are 10,000 near matches with $J(x, y) \in [.7, .9]$.

With signature length $r = 25$ and repetitions $t = 50$, hit probability for $J(x, y) = s$ is $1 - (1 - s^{25})^{50}$.
For example: Consider a database with 10,000,000 audio clips. You are given a clip \( x \) and want to find any \( y \) in the database with \( J(x, y) \geq .9 \).

- There are 10 true matches in the database with \( J(x, y) \geq .9 \).
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With signature length \( r = 25 \) and repetitions \( t = 50 \), hit probability for \( J(x, y) = s \) is \( 1 - (1 - s^{25})^{50} \).

- Hit probability for \( J(x, y) \geq .9 \) is \( \geq 1 - (1 - .9^{25})^{50} \approx .98 \)
- Hit probability for \( J(x, y) \in [.7, .9] \) is \( \leq 1 - (1 - .9^{25})^{50} \approx .98 \)
- Hit probability for \( J(x, y) \leq .7 \) is \( \leq 1 - (1 - .7^{25})^{50} \approx .007 \)
For example: Consider a database with 10,000,000 audio clips. You are given a clip $x$ and want to find any $y$ in the database with $J(x, y) \geq .9$.

- There are 10 true matches in the database with $J(x, y) \geq .9$.
- There are 10,000 near matches with $J(x, y) \in [.7, .9]$.

With signature length $r = 25$ and repetitions $t = 50$, hit probability for $J(x, y) = s$ is $1 - (1 - s^{25})^{50}$.

- Hit probability for $J(x, y) \geq .9$ is $\geq 1 - (1 - .9^{25})^{50} \approx .98$
- Hit probability for $J(x, y) \in [.7, .9]$ is $\leq 1 - (1 - .9^{25})^{50} \approx .98$
- Hit probability for $J(x, y) \leq .7$ is $\leq 1 - (1 - .7^{25})^{50} \approx .007$

Expected Number of Items Scanned: (proportional to query time)

$$\leq 10 + .98 \times 10,000 + .007 \times 9,989,990 \approx 80,000$$
**For example:** Consider a database with 10,000,000 audio clips. You are given a clip $x$ and want to find any $y$ in the database with $J(x, y) \geq .9$.

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With signature length $r = 25$ and repetitions $t = 50$, hit probability for $J(x, y) = s$ is $1 - (1 - s^{25})^{50}$.

- Hit probability for $J(x, y) \geq .9$ is $\geq 1 - (1 - .9^{25})^{50} \approx .98$
- Hit probability for $J(x, y) \in [.7, .9]$ is $\leq 1 - (1 - .9^{25})^{50} \approx .98$
- Hit probability for $J(x, y) \leq .7$ is $\leq 1 - (1 - .7^{25})^{50} \approx .007$

**Expected Number of Items Scanned:** (proportional to query time)

$$\leq 10 + .98 \times 10,000 + .007 \times 9,989,990 \approx 80,000 \ll 10,000,000.$$