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- Typically must compress the data on the fly, storing a data structure from which you can still learn useful information.
- Often the compression is randomized. E.g., bloom filters.
- Compared to traditional algorithm design, which focuses on minimizing runtime, the big question here is how much space is needed to answer queries of interest.
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SOME EXAMPLES

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Distinct Elements (Count-Distinct) Problem: Given a stream $x_1, \ldots, x_n$, output the number of distinct elements in the stream.
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Applications:

- Distinct IP addresses clicking on an ad or visiting a site.
- Number of distinct search engine queries.
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Google Sawzall, Facebook Presto, Apache Drill, Twitter Algebird
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Min-Hashing for Distinct Elements (variant of Flajolet-Martin):

- Let \( h : U \rightarrow [0, 1] \) be a random hash function (with a real valued output)
- \( s := 1 \)
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- Intuition: The larger \( d \) is, the smaller we expect \( s \) to be.
- Same idea as Flajolet-Martin algorithm and HyperLogLog, except they use discrete hash functions.
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- So estimate of \( \hat{d} = \frac{1}{s} - 1 \) output by the algorithm is correct if \( s \) exactly equals its expectation. \textbf{Does this mean } \mathbb{E}[\hat{d}] = d? \text{ No, but:}
- \textbf{Exercise:} Approximation is robust, i.e., if \(|s - \mathbb{E}[s]| \leq \epsilon \cdot \mathbb{E}[s] \) for any \( \epsilon \in (0, 1/2) \),

\[
(1 - 4\epsilon)d \leq \hat{d} \leq (1 + 4\epsilon)d
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So question is how well $s$ concentrates around its mean.

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$$\Pr \left[ |s - \mathbb{E}[s]| \geq \epsilon \mathbb{E}[s] \right] \leq \frac{\text{Var}[s]}{\epsilon^2 \mathbb{E}[s]^2} \leq \frac{1}{\epsilon^2}.$$

Bound is vacuous for any $\epsilon < 1$.

How can we improve accuracy?

- $s$: minimum of $d$ distinct hashes chosen randomly over $[0, 1]$, computed by hashing algorithm.
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**Analysis**

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**Chebyshev Inequality:**

\[ \Pr \left[ \left| d - \hat{d} \right| \geq 4\epsilon \cdot d \right] \leq \frac{\text{Var}[s]}{(\epsilon \mathbb{E}[s])^2} = \frac{\mathbb{E}[s]^2 / k}{\epsilon^2 \mathbb{E}[s]^2} = \frac{1}{k \cdot \epsilon^2} \]

\( s_j: \) minimum of \( d \) distinct hashes chosen randomly over \([0, 1]\). \( s = \frac{1}{k} \sum_{j=1}^{k} s_j. \) \( \hat{d} = \frac{1}{s} - 1: \) estimate of \( \# \) distinct elements \( d. \)
\( s = \frac{1}{k} \sum_{j=1}^{k} s_j \). Have already shown that for \( j = 1, \ldots, k \):

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\mathbb{E}[s_j] = \frac{1}{d+1} \implies \mathbb{E}[s] = \frac{1}{d+1} \quad \text{(linearity of expectation)}
\]

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How should we set $k$ if we want $4\epsilon \cdot d$ error with probability $\geq 1 - \delta$? $k = \frac{1}{\epsilon^2 \cdot \delta}$.

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Hashing for Distinct Elements:

- Let \( h_1, h_2, \ldots, h_k : U \rightarrow [0, 1] \) be random hash functions
- \( s_1, s_2, \ldots, s_k := 1 \)
- For \( i = 1, \ldots, n \)
  - For \( j=1, \ldots, k \), \( s_j := \min(s_j, h_j(x_i)) \)
- \( s := \frac{1}{k} \sum_{j=1}^{k} s_j \)
- Return \( \hat{d} = \frac{1}{s} - 1 \)

- Setting \( k = \frac{1}{\epsilon^2 \cdot \delta} \), algorithm returns \( \hat{d} \) with \( |d - \hat{d}| \leq 4\epsilon \cdot d \) with probability at least \( 1 - \delta \).
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![Diagram showing the process of hashing for distinct elements](image.png)
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- Space complexity is \( k = \frac{1}{\epsilon^2 \cdot \delta} \) real numbers \( s_1, \ldots, s_k \).
- \( \delta = 5\% \) failure rate gives a factor 20 overhead in space complexity.
How can we improve our dependence on the failure rate $\delta$?

The median trick: Run $t = O(\log \frac{1}{\delta})$ trials each with failure probability $\delta' = \frac{1}{4} - \frac{\epsilon^2}{4}$ hash functions.

- Letting $\hat{d}_1, \ldots, \hat{d}_t$ be the outcomes of the $t$ trials, return $\hat{d} = \text{median}(\hat{d}_1, \ldots, \hat{d}_t)$.
- If $> \frac{1}{2}$ of trials fall in $[(1 - \frac{4}{\epsilon}) \cdot \hat{d}, (1 + \frac{4}{\epsilon}) \cdot \hat{d}]$, then the median will.
How can we improve our dependence on the failure rate $\delta$?

**The median trick:** Run $t = \Theta(\log 1/\delta)$ trials each with failure probability $\delta' = 1/4$ – each using $k = \frac{1}{\delta' \epsilon^2} = \frac{4}{\epsilon^2}$ hash functions.
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![Diagram showing the median trick with points at $(1 - 4\varepsilon)d$, $d$, and $(1 + 4\varepsilon)d$.]
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THE MEDIAN TRICK

• \( \hat{d}_1, \ldots, \hat{d}_t \) are the outcomes of the \( t \) trials, each falling in

\[
[(1 - 4\epsilon)d, (1 + 4\epsilon)d]
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with probability at least \( 3/4 \). Let \( \hat{d} = \text{median}(\hat{d}_1, \ldots, \hat{d}_t) \).

What is the probability that the median \( \hat{d} \) falls in \( [(1 - 4\epsilon)d, (1 + 4\epsilon)d] \)?
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Apply Chernoff bound:
• \(\hat{d}_1, \ldots, \hat{d}_t\) are the outcomes of the \(t\) trials, each falling in

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\Pr\left(|X - \mathbb{E}[X]| \geq \frac{1}{3} \mathbb{E}[X]\right) \leq 2 \exp\left(-\frac{\frac{1}{3} \cdot \frac{3}{4} t}{2 + 1/3}\right) = e^{-\Theta(t)}.
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• Setting \(t = O(\log(1/\delta))\) gives failure probability \(e^{-\log(1/\delta)} = \delta\).
Upshot: The median of $t = O(\log(1/\delta))$ independent runs of the hashing algorithm for distinct elements returns

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**MEDIAN TRICK**
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**Total Space Complexity:** \( t \) trials, each using \( k = \frac{1}{\epsilon^2 \delta'} \) hash functions, for \( \delta' = 1/4 \). Space is \( \frac{4t}{\epsilon^2} = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right) \) real numbers (the minimum value of each hash function).
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No dependence on the number of distinct elements $d$ or the number of items in the stream $n$! Both can be very large.
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No dependence on the number of distinct elements $d$ or the number of items in the stream $n$! Both can be very large.

**A note on the median:** The median is often used as a robust alternative to the mean, when there are outliers (e.g., heavy tailed distributions, corrupted data).