COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor
Lecture 4
Last Class:

- 2-Level Hashing Analysis (linearity of expectation and Markov’s inequality)
- 2-universal and pairwise independent hash functions
- Chebyshev: \( \Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}[X]}{t^2} \)
LAST TIME

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This Time:

• Random hashing for load balancing. Motivating:
  • Stronger concentration inequalities: Chebyshev’s inequality, exponential tail bounds, and their connections to the law of large numbers.
  • The union bound.
Suppose random variable $X$ and can be written as

$$X = A_1 + A_2 + \ldots + A_n$$

where each $A_i$ are independent indicator variables with $\Pr(A_i) = p$. 

Note $\mathbb{E}[A_i] = p$ and $\text{Var}[A_i] = \mathbb{E}[A_i^2] - \mathbb{E}[A_i]^2 = p - p^2$.

By linearity of expectation and variance, $\mathbb{E}[X] = np$ and $\text{Var}[X] = np(1-p)$. 
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• Then, the distribution of $X$ is the **Binomial Distribution** and

$$\Pr[X = i] = \binom{n}{i} p^i (1 - p)^{n-i}$$
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$$E[X] = np \quad \text{Var}[X] = np(1 - p)$$
Randomized Load Balancing:

- $n$ requests randomly assigned to $k$ servers.

Mathematical Details:

Let $R_i$ be the number of requests assigned to the $i$th server. $R_i$ is binomial and has an expectation:

$$E[R_i] = n \sum_{j=1}^{k} Pr[j \text{ assigned to } i] = \frac{n}{k}.$$

The variance is:

$$\text{Var}[R_i] = \text{Var}[n \sum_{j=1}^{k} I[\text{request } j \text{ assigned to } i]] = n \sum_{j=1}^{k} \text{Var}[I[j \text{ assigned to } i]] = \frac{n}{k} \left(1 - \frac{1}{k^2}\right).$$
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- Variance:

$$\text{Var}[R_i] = \text{Var}\left[\sum_{j=1}^{n} I_{\text{request } j \text{ assigned to } i}\right] = \sum_{j=1}^{n} \text{Var}[I_j \text{ assigned to } i] = n \left(\frac{1}{k} - \frac{1}{k^2}\right).$$
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We want to upper bound:

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\Pr \left( \max_i (R_i) \geq \frac{2n}{k} \right) = \Pr \left( \left[ R_1 \geq \frac{2n}{k} \right] \text{ or } \ldots \text{ or } \left[ R_k \geq \frac{2n}{k} \right] \right) = \Pr \left( \bigcup_{i=1}^{k} \left[ R_i \geq \frac{2n}{k} \right] \right)
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$$= Pr\left(\bigcup_{i=1}^{k} \left[ R_i \geq \frac{2n}{k} \right]\right)$$

How do we do this since $R_1, \ldots, R_k$ are not independent?
THE UNION BOUND

Union Bound: For any random events $A_1, A_2, ..., A_k$,

$$\Pr(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \Pr(A_1) + \Pr(A_2) + \ldots + \Pr(A_k).$$
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On the first problem set, you will prove the union bound, as a consequence of Markov’s inequality.
APPLYING THE UNION BOUND

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[R_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

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As long as $k \ll \sqrt{n}$, the maximum server load will be small (compared to the expected load) with good probability.

\[n: \text{total number of requests, } k: \text{number of servers randomly assigned requests, } R_i: \text{number of requests assigned to server } i. \ \mathbb{E}[R_i] = \frac{n}{k}. \ \text{Var}[R_i] = \frac{n}{k}.\]
Pr(|X − E[X]| ≥ t) ≤ \frac{\text{Var}[X]}{t^2}

\textbf{X}: any random variable, \( t, s \): any fixed numbers.
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Why is this so powerful?

$X$: any random variable, $t$, $s$: any fixed numbers.
Consider drawing independent identically distributed (i.i.d.) random variables $X_1, \ldots, X_n$ with mean $\mu$ and variance $\sigma^2$. 
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How well does the sample average $S = \frac{1}{n} \sum_{i=1}^{n} X_i$ approximate the true mean $\mu$?

By Chebyshev's Inequality:

For any fixed value $\epsilon > 0$,

$$\Pr\left( |S - \mu| \geq \epsilon \right) \leq \frac{\text{Var}[S]}{\epsilon^2} = \frac{\sigma^2}{n \epsilon^2}.$$
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**Law of Large Numbers:**

With enough samples \( n \), the sample average will always concentrate to the mean \( \mu \).

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The number of servers must be small compared to the number of requests \((k = O(\sqrt{n}))\) for the maximum load to be bounded in comparison to the expected load with good probability.

\(n\): total number of requests, \(k\): number of servers randomly assigned requests.
The number of servers must be small compared to the number of requests ($k = O(\sqrt{n})$) for the maximum load to be bounded in comparison to the expected load with good probability.

- There are many requests routed to a relatively small number of servers so the load seen on each server is close to what is expected via law of large numbers.

$n$: total number of requests, $k$: number of servers randomly assigned requests.
Questions on union bound, Chebyshev’s inequality, random hashing?
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**Chebyshev’s:**

- \( \Pr(H \geq 60) \leq .25 \)
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$H$ has a simple Binomial distribution, so can compute these probabilities exactly.
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• Markov’s: \( \Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t} \). First Moment.
To be fair... Markov and Chebyshev’s inequalities apply much more generally than to Binomial random variables like coin flips.

Can we obtain tighter concentration bounds that still apply to very general distributions?

- **Markov’s**: \( \Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t} \). First Moment.
- **Chebyshev’s**: \( \Pr(|X - \mathbb{E}[X]| \geq t) = \Pr(|X - \mathbb{E}[X]|^2 \geq t^2) \leq \frac{\text{Var}[X]}{t^2} \). Second Moment.
To be fair... Markov and Chebyshev’s inequalities apply much more generally than to Binomial random variables like coin flips.

Can we obtain tighter concentration bounds that still apply to very general distributions?

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• What if we just apply Markov’s inequality to even higher moments?
Consider any random variable $X$:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right)$$
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$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right) \leq \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^4\right]}{t^4}.$$
A FOURTH MOMENT BOUND

Consider any random variable \( X \):

\[
\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right) \leq \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^4\right]}{t^4}.
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Consider any random variable $X$:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr \left( (X - \mathbb{E}[X])^4 \geq t^4 \right) \leq \frac{\mathbb{E} \left[ (X - \mathbb{E}[X])^4 \right]}{t^4}.$$ 

**Application to Coin Flips:** Recall: $n = 100$ independent fair coins, $H$ is the number of heads.

- Bound the fourth moment: 

$$
\sum_{i, j, k, \ell} c_{ijk\ell} \mathbb{E}[H_i H_j H_k H_\ell] = 1862.5
$$

where $H_i = 1$ if coin flip $i$ is heads and 0 otherwise. Then apply some messy calculations...
Consider any random variable $X$:

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where $H_i = 1$ if coin flip $i$ is heads and 0 otherwise.
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$$\mathbb{E} \left[ (H - \mathbb{E}[H])^4 \right] = \mathbb{E} \left[ \left( \sum_{i=1}^{100} H_i - 50 \right)^4 \right] = \sum_{i,j,k,\ell} c_{ijk\ell} \mathbb{E}[H_i H_j H_k H_\ell]$$

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- Apply Fourth Moment Bound: 

$$
\Pr\left(|H - \mathbb{E}[H]| \geq t\right) \leq \frac{1862.5}{t^4}.
$$
### Tighter Bounds

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<thead>
<tr>
<th>Chebyshev’s:</th>
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- We aren’t restricted to applying Markov’s to $|X - \mathbb{E}[X]|^k$ for some $k$. Can apply to any monotonic function $f(|X - \mathbb{E}[X]|)$. 

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- We aren’t restricted to applying Markov’s to \(|X - E[X]|^k\) for some \(k\). Can apply to any monotonic function \(f(|X - E[X]|)\).
- **Why monotonic?** \(Pr(|X - E[X]| > t) = Pr(f(|X - E[X]|) > f(t))\).
• **Moment Generating Function:** Consider for any \( r > 0 \):

\[
M_r(X) = e^{r \cdot (X - \mathbb{E}[X])} = \sum_{k=0}^{\infty} \frac{r^k (X - \mathbb{E}[X])^k}{k!}
\]

and note \( M_r(X) \) is monotonic for any \( r > 0 \)
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$$\Pr[|X - \mathbb{E}[X]| \geq \lambda] = \Pr[M_r(X) \geq e^{r \lambda}] \leq \frac{\mathbb{E}[M_r(X)]}{e^{r \lambda}}$$
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• Weighted sum of all moments ($r$ controls the weights) and choosing $r$ appropriately lets one prove a number of very powerful exponential concentration bounds such as Chernoff, Bernstein, Hoeffding, Azuma, Berry-Esseen, etc.
Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n \in [-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^{n} X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^{n} X_i]$. For any $t \geq 0$:

$$
\Pr\left(\left|\sum_{i=1}^{n} X_i - \mu\right| \geq t\right) \leq 2 \exp\left(-\frac{t^2}{2\sigma^2 + \frac{4}{3}Mt}\right).
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Assume that $M = 1$ and plug in $t = s \cdot \sigma$ for $s \leq \sigma$. 
Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n \in [-1,1]$. Let $\mu = \mathbb{E}[\sum_{i=1}^{n} X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^{n} X_i]$. For any $s \geq 0$:

$$\Pr \left( \left| \sum_{i=1}^{n} X_i - \mu \right| \geq s\sigma \right) \leq 2 \exp \left( -\frac{s^2}{4} \right).$$

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Compare to Chebyshev's: $\Pr \left( \left| \sum_{i=1}^n X_i - \mu \right| \geq s \sigma \right) \leq \frac{1}{s^2}$. 

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**Compare to Chebyshev’s:** $\Pr \left( \left| \sum_{i=1}^{n} X_i - \mu \right| \geq s \sigma \right) \leq \frac{1}{s^2}$.

• An exponentially stronger dependence on $s$!
Consider again bounding the number of heads $H$ in $n = 100$ independent coin flips.

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$H$: total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$. 
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Getting much closer to the true probability.

$H$: total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$. 
Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n$ all falling in $[-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \text{Var}[X_i]$. For any $t \geq 0$:

$$\Pr \left( \left| \sum_{i=1}^n X_i - \mu \right| \geq t \right) \leq 2 \exp \left( -\frac{t^2}{2\sigma^2 + \frac{4}{3}Mt} \right).$$

A useful variation for binary (indicator) random variables is:

Chernoff Bound (simplified version): Consider independent random variables $X_1, \ldots, X_n$ taking values in $\{0, 1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$. For any $\delta \geq 0$

$$\Pr \left( \left| \sum_{i=1}^n X_i - \mu \right| \geq \delta \mu \right) \leq 2 \exp \left( -\frac{\delta^2 \mu}{2 + \delta} \right).$$