COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor
Lecture 12
CENTRAL LIMIT THEOREM
Bernstein Inequality (Simplified): Consider independent random variables $X_1, \ldots, X_n$ falling in $[-1,1]$. Let $\mu = \mathbb{E}\left[\sum X_i\right]$, $\sigma^2 = \text{Var}\left[\sum X_i\right]$, and $s \leq \sigma$. Then:

$$\Pr\left(\left|\sum_{i=1}^{n} X_i - \mu\right| \geq s\sigma\right) \leq 2 \exp\left(-\frac{s^2}{4}\right).$$
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$$\Pr \left( \left| \sum_{i=1}^{n} X_i - \mu \right| \geq s\sigma \right) \leq 2\exp\left( -\frac{s^2}{4} \right).$$

Can plot this bound for different $s$: 
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![Plot](image)

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**Exercise:** Using this can show that for $X \sim \mathcal{N}(0, \sigma^2)$: for any $s \geq 0$,

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Essentially the same bound that Bernstein’s inequality gives!
GAUSSIAN TAILS

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**Central Limit Theorem Interpretation:** Bernstein’s inequality gives a quantitative version of the CLT. The distribution of the sum of *bounded* independent random variables can be upper bounded with a Gaussian (normal) distribution.
Central Limit Theorem

**Stronger Central Limit Theorem:** The distribution of the sum of $n$ *bounded* independent random variables converges to a Gaussian (normal) distribution as $n$ goes to infinity.
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- Why is the Gaussian distribution so important in statistics, science, ML, etc.?
**Stronger Central Limit Theorem:** The distribution of the sum of $n$ *bounded* independent random variables converges to a Gaussian (normal) distribution as $n$ goes to infinity.

- Why is the Gaussian distribution so important in statistics, science, ML, etc.?
- Many random variables can be approximated as the sum of a large number of small and roughly independent random effects. Thus, their distribution looks Gaussian by CLT.
SUMMARY OF FIRST SECTION
WHAT WE’VE COVERED

- **Probability Tools**: Linearity of Expectation, Linear of Variance of Independent Variables, Concentration Bounds (Markov, Chebyshev, Bernstein, Chernoff), Union Bound, Median Trick.

- **Hash Tables and Bloom Filters**: Analyzing collisions. Building 2-level hash tables. Bloom filters and false positive rates.

- **Locality Sensitive Hashing**: MinHash for Jaccard Similarity, SimHash for Cosine Similarity. Nearest Neighbor. All-Pairs Similarity Search.

- **Small Space Data Stream Algorithms**: a) distinct items, b) frequent elements, c) frequent moments (homework).

- **Johnson Lindenstrauss Lemma**: Reducing dimension of vectors via random projection such that pairwise distances are approximately preserved. Application to clustering.
Randomization is an important tool in working with large datasets.

Lets us solve ‘easy’ problems that get really difficult on massive datasets. Fast/space efficient look up (hash tables and bloom filters), distinct items counting, frequent items counting, near neighbor search (LSH), etc.

The analysis of randomized algorithms sometimes leads to complex output distributions, which we can’t compute exactly. We use concentration inequalities to bound these distributions and behaviors like accuracy, space usage, and runtime.

Concentration inequalities and probability tools used in randomized algorithms are also fundamental in statistics, machine learning theory, probabilistic modeling of complex systems, etc.
• Linearity of Expectation: For any random variables $X_1, \ldots, X_n$ and constants $c_1, \ldots, c_n$,

$$\mathbb{E}[c_1 X_1 + \ldots + c_n X_n] = c_1 \mathbb{E}[X_1] + \ldots + c_n \mathbb{E}[X_n]$$
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• **Independent Random Variables:** $X_1, X_2, \ldots, X_n$ are independent random variables if for any set $S \subset [n]$ and values $a_1, a_2, \ldots, a_n$

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\Pr(X_i = a_i \text{ for all } i \in S) = \prod_{i \in S} \Pr(X_i = a_i)
$$

They are *$k$-wise independent* if this holds for $S$ with $|S| \leq k$. 
USEFUL PROBABILITY FACTS (1/2)

- **Linearity of Expectation:** For any random variables $X_1, \ldots, X_n$ and constants $c_1, \ldots, c_n$,

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- **Linearity of Variance:** If $X_1, \ldots, X_n$ are independent (in fact 2-wise independent suffices) then for any constants $c_1, \ldots, c_n$

  $$\text{Var}[c_1 X_1 + \ldots + c_n X_n] = c_1^2 \text{Var}[X_1] + \ldots + c_n^2 \text{Var}[X_n]$$
• **Union Bound**: For any events $A_1, A_2, A_3, \ldots$

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\Pr[\text{at least one of the events happens}] = \Pr \left[ \bigcup_i A_i \right] \leq \sum_i \Pr[A_i].
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• An **indicator random variable** $X$ just takes the values 0 or 1:

$$\mathbb{E}[X] = p \quad \text{Var}[X] = p(1 - p) \quad \text{where } p = \Pr[X = 1]$$
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• If $Y = X_1 + \ldots + X_n$ where each $X_i$ are independent and

\[
p = \Pr[X_1 = 1] = \ldots = \Pr[X_n = 1]
\]

then $Y$ is a *binomial random variable*. Using linearity of expectation and variance,

\[
\mathbb{E}[Y] = np \quad \text{Var}[Y] = np(1 - p)
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Let $R_i$ be number of balls bin $i$. Then $R_i \sim \text{Bin}(n, \frac{1}{m})$ and $\mathbb{E}[R_i] = \frac{n}{m}$, $\text{Var}[R_i] = \frac{n}{m} \cdot (1 - \frac{1}{m})$. $R_i$ and $R_j$ not independent!
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- Union Bound implies $\Pr[\max(R_1, \ldots, R_m) > t] \leq \sum_i \Pr[R_i > t]$

- In the exam, you’ll be expected to do calculations like these.
**BALLS AND BINS (1/2)**

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- Union Bound implies $\Pr[\text{max}(R_1, \ldots, R_m) > t] \leq \sum_i \Pr[R_i > t]$
- $\Pr[\text{no collisions}] = \frac{m-1}{m} \cdot \frac{m-2}{m} \cdot \ldots \cdot \frac{m-(n-1)}{m}$

$$\Pr[\text{collisions}] = \Pr[\text{max}(R_1, \ldots, R_m) > 1] \leq 1/8 \text{ if } m > 4n^2$$

and more generally

$$\Pr[\text{max}(R_1, \ldots, R_m) \geq 2n/m] \leq m^2/n$$
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Let \( R_i \) be number of balls bin \( i \). Then \( R_i \sim \text{Bin}(n, \frac{1}{m}) \) and \( \mathbb{E}[R_i] = \frac{n}{m} \), \( \text{Var}[R_i] = \frac{n}{m} \cdot (1 - \frac{1}{m}) \). \( R_i \) and \( R_j \) not independent!

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In the exam, you’ll be expected to do calculations like these.
Let $T$ be the number of bins where $R_i = 0$. We showed:

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• The probability the next $k$ balls thrown all land in non-empty bins is

$$(1 - T/m)^k$$

and this lets us analyze the false positive rate of a Bloom filter.
• Hash function $h : U \rightarrow [n]$ is two universal if:

$$\Pr[h(x) = h(y)] \leq \frac{1}{n} \quad \text{for all } x \neq y \in U$$
• Hash function \( h : U \rightarrow [n] \) is **two universal** if:

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• Hash function \( h : U \rightarrow [n] \) is **\( k \)-wise independent** if \( \{h(e)\}_{e \in U} \) are \( k \)-wise independent and each \( h(e) \) is uniform in \([n]\).
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• Hash function \( h : U \rightarrow [n] \) is fully independent if \( \{h(e)\}_{e \in U} \) are independent and each \( h(e) \) is uniform in \([n]\).
THREE MAIN CONCENTRATION BOUNDS

- **Markov.** For any non-negative random variable $X$ and $t > 0$,
  \[ \Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t} . \]
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- **Chebyshev.** For any random variable $X$ and $t > 0$, 
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  \Pr[X \geq t + \mathbb{E}[X]] \leq \Pr[|X - \mathbb{E}[X]| \geq t] \leq \frac{\text{Var}[X]}{t^2}.
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• **Chernoff.** Let $X_1, \ldots, X_n$ be independent $\{0, 1\}$ random variables with $\mu = \mathbb{E}[\sum_i X_i]$. Then for any $\delta > 0$,

$$\Pr[|(\sum_i X_i) - \mu| \geq \delta \mu] \leq 2 \exp \left(-\frac{\delta^2 \mu}{\delta + 2}\right).$$
THREE MAIN CONCENTRATION BOUNDS

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  \Pr[X \geq t] \leq \mathbb{E}[X]/t.
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- Generally, Chernoff gives better results than Chebyshev and Chebyshev gives better results than Markov. So choose bound based on how much you know about $X$. 

Bernstein generalizes Chernoff to arbitrary bounded $X_i$ variables.
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• **Chebyshev.** For any random variable $X$ and $t > 0$,

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• **Bernstein** generalizes Chernoff to arbitrary bounded $X_i$ variables.
• Want to learn a quantity $q$. Suppose you have a randomized algorithm that returns $X$ that has expectation $q$ and variance $\sigma^2$.

\[ \text{Median Trick: Let } t = t_1 t_2 \text{ where } t_1 = 4 \frac{\sigma^2}{\epsilon^2 q^2} \text{ and } t_2 = O(\log \frac{1}{\delta}). \] Let $A_1$ be average of first $t_1$ results, let $A_2$ be average of next $t_1$ results etc. Then, $\Pr[|A_i - q| \geq \epsilon q] \leq \frac{1}{4}$ and $\Pr[|\text{median}(A_1, \ldots, A_{t_2}) - q| \geq \epsilon q] \leq \delta$. 

• Want to learn a quantity $q$. Suppose you have a randomized algorithm that returns $X$ that has expectation $q$ and variance $\sigma^2$.

• To get a good estimate of $q$, repeat algorithm $t$ times to get $X_1, \ldots, X_t$ and let $A = (X_1 + \ldots + X_t)/t$. Then, if $t = \frac{\sigma^2}{\delta \epsilon^2 q^2}$

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AVERAGING AND THE MEDIAN TRICK

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2-LEVEL HASH TABLES VS. BLOOM FILTER

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• Bloom Filter:
  • Does not actually store the items in $S$, just a binary array from which we make various deductions.
  • Uses only $O(|S|)$ space but at the cost of sometimes answering “yes” when answer should be “no” (a false positive).
  • If the Bloom Filter array is length $m$, false positive probability is roughly $\left(1 - e^{-k|S|/m}\right)^k$ where $k$ is the number of hash functions used. Picking $k = \ln 2 \cdot m/|S|$ gives probability $1/2^{(\ln 2) m/|S|}$.

Also saw stacked hash tables in the homework.
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• Also saw stacked hash tables in the homework.
• Designed a hash function for hashing sets such that for sets $A$ and $B$, 
\[ \Pr[MH(A) = MH(B)] = J(A, B) = \frac{|A \cap B|}{|A \cup B|}. \]

\[
MH(A) = \min_{x \in A} h(x) \quad \text{where} \quad h : U \to [0, 1] \text{ is fully independent}
\]
• Designed a hash function for hashing sets such that for sets $A$ and $B$, \( \Pr[\text{MH}(A) = \text{MH}(B)] = J(A, B) = \frac{|A \cap B|}{|A \cup B|} \).

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• Can form signature of set $A$ using $r$ independent hash functions:

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\text{signature}(A) = (\text{MH}_1(A), \ldots, \text{MH}_r(A))
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$$\text{signature}_1(A), \ldots, \text{signature}_t(A).$$

Then if $s = J(A, B)$,

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• To find all pairs of similar sets amongst $A_1, A_2, A_3, \ldots$ only compare a pair if there exists $i$, their $i$th signatures match.
• We want to compute something about the stream $x_1, x_2, \ldots, x_m$ with only one pass over the stream and limited space.
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  • Distinct Items: Can estimate $D = |\{i : f_i > 0\}$ up to a factor $1 + \epsilon$ with probability $1 - \delta$ in $O(\epsilon^{-2} \log 1/\delta)$ space. Main idea was exploiting the fact the expected value of the minimum of $d$ number picked randomly in $[0, 1]$ is $1/(d + 1)$. 

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  • Frequently Elements Items: Can return a set $S$ such that:

\[
  f_i \geq \frac{m}{k} \text{ implies } i \in S \quad \text{and} \quad i \in S \text{ implies } f_i \geq \frac{m(1 - \epsilon)}{k}
\]

with probability $1 - \delta$ in $O(k/\epsilon \cdot \log 1/\delta)$ space.
• Sum of Powers: In the homework we considered estimating quantities such as $\sum f_i^k$. 
Count-Min Sketch: A random hashing based method closely related to bloom filters.
Count-Min Sketch: A random hashing based method closely related to bloom filters.

random hash function $h$

$m$ length array $A$
Count-Min Sketch: A random hashing based method closely related to bloom filters.
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Use $A[h(x)]$ to estimate $f(x)$, the frequency of $x$ in the stream.

- **Claim:** $A[h(x)] \geq f(x)$.
- **Claim:** $A[h(x)] \leq f(x) + 2n/m$ with probability at least $1/2$. 
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How can we increase this probability to $1 - \delta$ for arbitrary $\delta > 0$?
• Estimate $f(x)$ with $\tilde{f}(x) = \min_{i \in [t]} A_i[h_i(x)]$.

• Then $\Pr[f(x) \leq \tilde{f}(x) \leq f(x) + 2n/m] \geq 1 - \frac{1}{2^t}$.

• Setting $t = \log(1/\delta)$ ensures probability is at least $1 - \delta$.

• Setting $m = 2^{k/\epsilon}$ ensures $2n/m = \epsilon n/k$ and that's enough to determine whether we need to output the element.
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• Then $\Pr[f(x) - \tilde{f}(x) \leq f(x) + 2\frac{n}{m}] \geq 1 - \frac{1}{2^t}$.

• Setting $t = \log(1/\delta)$ ensures probability is at least $1 - \delta$.

• Setting $m = 2^{k/\epsilon}$ ensures $2\frac{n}{m} = \epsilon$ and that's enough to determine whether we need to output the element.
### Count-Min Sketch Accuracy

- Estimate \( f(x) \) with \( \tilde{f}(x) = \min_{i \in [t]} A_i[h_i(x)] \).
- Then \( \Pr[f(x) \leq \tilde{f}(x) \leq f(x) + 2n/m] \geq 1 - 1/2^t \).
- Setting \( t = \log(1/\delta) \) ensures probability is at least \( 1 - \delta \).
- Setting \( m = 2^k/\epsilon \) ensures \( 2n/m = \epsilon n/k \) and that's enough to determine whether we need to output the element.

![Diagram of Count-Min Sketch](image)

<table>
<thead>
<tr>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( \cdots )</th>
<th>( A_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 5 1 0 6 12 104 1 3 4</td>
<td>1 6 1 10 78 80 4 11 3 5</td>
<td></td>
<td>90 1 52 6 3 12 33 9 3 2</td>
</tr>
</tbody>
</table>

**t random hash functions**

- \( h_1, h_2, \ldots, h_t \)
**Count-Min Sketch Accuracy**

- Estimate $f(x)$ with $\tilde{f}(x) = \min_{i \in [t]} A_i[h_i(x)]$.
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- Setting $t = \log(1/\delta)$ ensures probability is at least $1 - \delta$.
- Setting $m = \frac{2^k}{\epsilon}$ ensures $\frac{2n}{m} = \frac{\epsilon n}{k}$ and that's enough to determine whether we need to output the element.

![Diagram showing t random hash functions $h_1, h_2, \ldots, h_t$ and t length m arrays $A_1, A_2, \ldots, A_t$.]

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<th>$A_1$</th>
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<th>$A_t$</th>
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<tr>
<td>2  5</td>
<td>1  6</td>
<td>90 1</td>
</tr>
<tr>
<td>1  0</td>
<td>1  10</td>
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</tr>
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count-min sketch accuracy

- Estimate $f(x)$ with $\tilde{f}(x) = \min_{i \in \{1, \ldots, t\}} A_i[h_i(x)]$.

- Then $\Pr[f(x) \leq \tilde{f}(x) \leq f(x) + 2n/m] \geq 1 - \frac{1}{2^t}$.

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\textbf{COUNT-MIN SKETCH ACCURACY}

- Estimate $f(x)$ with $\tilde{f}(x) = \min_{i \in [t]} A_i[h_i(x)]$.
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\begin{itemize}
  \item Setting $t = \log(1/\delta)$ ensures probability is at least $1 - \delta$.
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Johnson Lindenstrauss Lemma: If $\mathbf{M} \in \mathbb{R}^{m \times d}$ is a random matrix with $m = O \left( \epsilon^{-2} \log n \right)$, for $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ with high probability, for all $i, j$:

$$(1 - \epsilon) \| \vec{x}_i - \vec{x}_j \|_2 \leq \| \mathbf{M} \vec{x}_i - \mathbf{M} \vec{x}_j \|_2 \leq (1 + \epsilon) \| \vec{x}_i - \vec{x}_j \|_2$$

where $\| \vec{z} \|_2^2$ is the sum of squared entries of $\vec{z}$. 
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Johnson Lindenstrauss Lemma: If $M \in \mathbb{R}^{m \times d}$ is a random matrix with $m = O(\epsilon^{-2} \log n)$, for $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ with high probability, for all $i, j$:

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Proof Idea:

- Follows from Distributional JL: If $M \in \mathbb{R}^{m \times d}$ has $\mathcal{N}(0, 1/m)$ entries where $m = O(\epsilon^{-2} \log(1/\delta))$ then for any $\vec{y} \in \mathbb{R}^d$, $\|M\vec{y}\|_2 \approx \|\vec{y}\|_2$ with probability at least $1 - \delta$. 
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- To prove Distributional JL Lemma:
  - By linearity of expectation and variance, $\mathbb{E}[\|M\vec{y}\|_2^2] = \|\vec{y}\|_2^2$.
  - $\|M\vec{y}\|_2^2$ is the sum of $m$ squared independent normal distributions and is tightly concentrated around the expectation.
EXTRA SLIDE
Our algorithm uses continuous valued fully random hash functions.
Our algorithm uses continuous valued fully random hash functions. Can't be implemented...

• The idea of using the minimum hash value of $x_1, \ldots, x_n$ to estimate the number of distinct elements naturally extends to when the hash functions map to discrete values.

• Flajolet-Martin (LogLog) algorithm and HyperLogLog. Estimate # distinct elements based on maximum number of trailing zeros $m$. The more distinct hashes we see, the higher we expect this maximum to be.
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Estimate # distinct elements based on maximum number of trailing zeros $m$. 
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Estimate # distinct elements based on maximum number of trailing zeros $m$.

The more distinct hashes we see, the higher we expect this maximum to be.
Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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Estimate \# distinct elements based on maximum number of trailing zeros \( m \).
Flajolet-Martin (LogLog) algorithm and HyperLogLog.

| $h(x_1)$ | 1010010 |
| $h(x_2)$ | 1001100 |
| $h(x_3)$ | 1001110 |
| $\vdots$ | $\vdots$ |
| $h(x_n)$ | 1011000 |

Estimate $\#$ distinct elements based on maximum number of trailing zeros $m$.

With $d$ distinct elements, roughly what do we expect $m$ to be?

a) $O(1)$  
   b) $O(\log d)$  
   c) $O(\sqrt{d})$  
   d) $O(d)$
Flajolet-Martin (LogLog) algorithm and HyperLogLog.

Estimate \# distinct elements based on maximum number of trailing zeros \( m \).

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\[
\Pr(h(x_i) \text{ has } x \text{ trailing zeros}) = \]

\[
\text{Total Space: } O(\log \log d \epsilon^2 + \log d) \text{ for an } \epsilon \text{ approximate count.}
\]

Note: Careful averaging of estimates from multiple hash functions.
LOGLOG COUNTING OF DISTINCT ELEMENTS

Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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```

Estimate \# distinct elements based on maximum number of trailing zeros \( m \).

With \( d \) distinct elements, roughly what do we expect \( m \) to be?

\[
Pr(h(x_i) \text{ has } x \text{ trailing zeros}) = \frac{1}{2^x}
\]

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Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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Estimate \# distinct elements based on maximum number of trailing zeros $m$.

With $d$ distinct elements, roughly what do we expect $m$ to be?

$$\Pr(h(x_i) \text{ has } \log d \text{ trailing zeros}) = \frac{1}{2^\log d}$$
Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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- Given data structures (sketches) \(HLL(x_1, \ldots, x_n), HLL(y_1, \ldots, y_n)\) it is easy to merge them to give \(HLL(x_1, \ldots, x_n, y_1, \ldots, y_n)\).

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- Set the maximum \# of trailing zeros to the maximum in the two sketches.

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