COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor
Lecture 5
EXPONENTIAL CONCENTRATION BOUNDS

• Can sometimes get tighter bounds than Markov via:

\[ \Pr[|X - \mathbb{E}[X]| \geq \lambda] = \Pr[|X - \mathbb{E}[X]|^k \geq \lambda^k] \leq \frac{\mathbb{E}[|X - \mathbb{E}[X]|^k]}{\lambda^k} \]

• **Moment Generating Function:** Consider for any \( t > 0 \):

\[ M_t(X) = e^{t \cdot (X - \mathbb{E}[X])} \]

and note \( M_t(X) \) is monotonic for any \( t > 0 \)
EXponential Concentration Bounds

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M_t(X) = e^{t \cdot (X - \mathbb{E}[X])} = \sum_{k=0}^{\infty} \frac{t^k(X - \mathbb{E}[X])^k}{k!}
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Expontential concentration bounds

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- Weighted sum of all moments (\( t \) controls the weights) and choosing \( t \) appropriately lets one prove a number of very powerful **exponential concentration bounds** such as Chernoff, Bernstein, Hoeffding, Azuma, Berry-Esseen, etc.
**Bernstein Inequality:** Consider independent random variables $X_1, \ldots, X_n$ all falling in $[-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^{n} X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \text{Var}[X_i]$. For any $t \geq 0$:

$$
\Pr \left( \left| \sum_{i=1}^{n} X_i - \mu \right| \geq t \right) \leq 2 \exp \left( - \frac{t^2}{2\sigma^2 + \frac{4}{3}Mt} \right).
$$

Assume that $M = 1$ and plug in $t = s \cdot \sigma$ for $s \leq \sigma$. Compare to Chebyshev’s:

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\Pr \left( \left| \sum_{i=1}^{n} X_i - \mu \right| \geq s \cdot \sigma \right) \leq \frac{1}{s^2}.
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• An exponentially stronger dependence on $s^2$!
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$$\Pr \left( \left| \sum_{i=1}^n X_i - \mu \right| \geq s\sigma \right) \leq 2 \exp \left( -\frac{s^2}{4} \right).$$

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- An exponentially stronger dependence on $s$!
Consider again bounding the number of heads $H$ in $n = 100$ independent coin flips.

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Getting much closer to the true probability.

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**Solution:** Bloom filters (repeated random hashing). Will use much less space than a hash table.
Chose $k$ independent random hash functions $h_1, \ldots, h_k$ mapping the universe of elements $U \rightarrow [m]$. 

- Maintain an array $A$ containing $m$ bits, all initially 0.
- insert $(x)$: set all bits $A[h_1(x)] = \ldots = A[h_k(x)] := 1$.
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No false negatives. False positives more likely with more insertions.
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m bit array $A$

0 0 0 0 0 0 0 0 0 0 0 0
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Insertions:

$$
\begin{array}{c}
\text{m bit array } A \\
1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0
\end{array}
$$

Queries:
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APPLICATIONS: CACHING

Akamai (Boston-based company serving 15 – 30% of all web traffic) applies bloom filters to prevent caching of ‘one-hit-wonders’ – pages only visited once fill over 75% of cache.

![Graph showing disk writes per second with bloom filter turned on](image_url)
AKAMAI (Boston-based company serving 15 – 30% of all web traffic) applies bloom filters to prevent caching of ‘one-hit-wonders’ – pages only visited once fill over 75% of cache.

- When url $x$ comes in, if $\text{query}(x) = 1$, cache the page at $x$. If not, run $\text{insert}(x)$ so that if it comes in again, it will be cached.
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- When url $x$ comes in, if $query(x) = 1$, cache the page at $x$. If not, run $insert(x)$ so that if it comes in again, it will be cached.

- **False positive:** A new url (possible one-hit-wonder) is cached. If the bloom filter has a false positive rate of $\delta = .05$, the number of cached one-hit-wonders will be reduced by at least 95%.
For a bloom filter with $m$ bits and $k$ hash functions, the insertion and query time is $O(k)$. 

How does the false positive rate $\delta$ depend on $m$, $k$, and the number of items inserted?

Step 1: What is the probability that after inserting $n$ elements, the $i$th bit of the array $A$ is still 0?

$n \times k$ total hashes must not hit bit $i$.

$\Pr(A[i] = 0) = \Pr(h_1(x_1) \neq i \cap ... \cap h_k(x_1) \neq i \cap h_1(x_2) \neq i \cap ...)$

$= \Pr(h_1(x_1) \neq i) \times ... \times \Pr(h_k(x_1) \neq i) \times \Pr(h_1(x_2) \neq i) ...$

$= (1 - \frac{1}{m})^{kn}$
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$$= \Pr(h_1(x_1) \neq i) \times \ldots \times \Pr(h_k(x_1) \neq i) \times \Pr(h_1(x_2) \neq i) \ldots$$

$k \cdot n$ events each occurring with probability $1 - 1/m$
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$n$: total number items in filter, $m$: number of bits in filter, $k$: number of random hash functions, $h_1, \ldots, h_k$: hash functions, $A$: bit array, $\delta$: false positive rate.
How does the false positive rate $\delta$ depend on $m$, $k$, and the number of items inserted?

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$$\Pr(A[i] = 0) = \left(1 - \frac{1}{m}\right)^{kn} \approx e^{-\frac{kn}{m}}$$

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Let $T$ be the number of zeros in the array after $n$ inserts. Then,

$$E[T] = m \left(1 - \frac{1}{m}\right)^{kn} \approx me^{-\frac{kn}{m}}$$

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If $T$ is the number of 0 entries, for a non-inserted element $w$:

$$\Pr(A[h_1(w)] = \ldots = A[h_k(w)] = 1)$$

$$= \Pr(A[h_1(w)] = 1) \times \ldots \times \Pr(A[h_k(w)] = 1)$$

$$= (1 - T/m) \times \ldots \times (1 - T/m)$$

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• How small is $T/m$? Note that $\frac{T}{m} \geq \frac{m-nk}{m} \approx e^{-kn/m}$ when $kn \ll m$. More generally, it can be shown that $T/m = \Omega \left( e^{-kn/m} \right)$ via Theorem 2 of:

cglab.ca/~morin/publications/ds/bloom-submitted.pdf
**False Positive Rate:** with \( m \) bits of storage, \( k \) hash functions, and \( n \) items inserted

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• Balances between filling up the array with too many hashes and having enough hashes so that even when the array is pretty full, a new item is unlikely to have all its bits set (yield a false positive).
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![Graph showing the false positive rate as a function of the number of hash functions $k$. The graph shows an increase in the false positive rate as $k$ increases.](image-url)
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Stream Processing: Have a massive dataset $X$ with $n$ items $x_1, x_2, \ldots, x_n$ that arrive in a continuous stream. Not nearly enough space to store all the items (in a single location).

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- Compared to traditional algorithm design, which focuses on minimizing runtime, the big question here is how much space is needed to answer queries of interest.
- **Sensor data:** images from telescopes (15 terabytes per night from the Large Synoptic Survey Telescope), readings from seismometer arrays monitoring and predicting earthquake activity, traffic cameras and travel time sensors (Smart Cities), electrical grid monitoring.
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Google Sawzall, Facebook Presto, Apache Drill, Twitter Algebird