COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Andrew McGregor
Lecture 2
Today:

- Investigate linearity of expectation and variance.
- Algorithmic application of linearity of expectation and variance.
- Introduce Markov’s inequality, a fundamental concentration bound, that let us prove that a random variable lies close to its expectation with good probability.
- Learn about random hash functions, which are a key tool in randomized methods for data processing. Probabilistic analysis via linearity of expectation.
• **Expectation:** \( \mathbb{E}[X] = \sum_{s \in S} \Pr(X = s) \cdot s \).
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• **Variance:** \( \text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]. \)
SOME PROBABILITY REVIEW

- **Expectation:** \( \mathbb{E}[X] = \sum_{s \in S} \Pr(X = s) \cdot s \).
- **Variance:** \( \text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] \).
- **Two random variables** \( X, Y \) **are independent** if for all \( s, t \), \( \{X = s\} \) and \( \{Y = t\} \) are independent events. In other words:
  \[
  \Pr(\{X = s\} \cap \{Y = t\}) = \Pr(X = s) \cdot \Pr(Y = t).
  \]
When are the expectation and variance linear?

I.e., under what conditions on \( X \) and \( Y \) do we have:

\[
\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]
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and

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\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y].
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and

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y].$$

Last time we showed that linearity of expectation is true regardless of whether the random variables were independent.

$X, Y$: any two random variables.
**LINEARITY OF VARIANCE**

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- They claim that they have a database of 1,000,000 unique CAPTCHAS. A random one is chosen for each security check.
- You want to independently verify this claimed database size.
- You could make test checks until you see 1,000,000 unique CAPTCHAS: would take $\geq 1,000,000$ checks!
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Note that if the same CAPTCHA shows up four times this counts as \( \binom{4}{2} \) duplicates.
Let $D_{i,j} = 1$ if tests $i$ and $j$ give the same CAPTCHA, and 0 otherwise. An indicator random variable.

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Let $D_{i,j} = 1$ if tests $i$ and $j$ give the same CAPTCHA, and 0 otherwise. An **indicator random variable**. The number of pairwise duplicates (a random variable) is:

$$D = \sum_{i,j \in [m], i < j} D_{i,j}.$$

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$$E[D] = \sum_{i,j\in[m],i<j} E[D_{i,j}].$$

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$$\mathbb{E}[D] = \sum_{i,j \in [m], i < j} \frac{1}{n} = \frac{(m)}{2} \frac{n}{n} = \frac{m(m-1)}{2n}.$$
You take \( m = 1000 \) samples. If the database size is as claimed \( (n = 1,000,000) \) then expected number of duplicates is:

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\mathbb{E}[D] = \frac{m(m - 1)}{2n} = .4995
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**Concentration Inequalities:** Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.

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MARKOV’S INEQUALITY

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The larger the deviation $t$, the smaller the probability.
Expected number of duplicate CAPTCHAS:

$$E[D] = \frac{m(m-1)}{2n} = .4995.$$

You see $D = 10$ duplicates.

$n$: number of CAPTCHAS in database ($n = 1000000$ claimed), $m$: number of random CAPTCHAS drawn to check database size ($m = 1000$ in this example), $D$: number of pairwise duplicates in $m$ random CAPTCHAS.
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Applying Markov’s inequality, if the real database size is \( n = 1000000 \) the probability of this happening is:

\[ \Pr[D \geq 10] \leq \frac{\mathbb{E}[D]}{10} = \frac{.4995}{10} \approx .05 \]

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This is pretty small and you feel pretty sure the number of unique CAPTCHAS is much less than 1000000.

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**Classic Solution:** Hash tables

- *Static hashing* since we won’t worry about insertion and deletion today.
• **hash function** \( h : U \rightarrow [n] \) maps elements from the universe to indices \( 1, \ldots, n \) of an array.
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• **Collisions:** when we insert $m$ items into the hash table we may have to store multiple items in the same location (typically as a linked list).
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How Can We Bound $c$?

- In the worst case, could have $c = m$ (all items hash to the same location). In the best case, $c \approx m/n$. 
Let $h : U \rightarrow [n]$ be a random hash function.

- I.e., for $x \in U$, $\Pr(h(x) = i) = \frac{1}{n}$ for all $i = 1, \ldots, n$ and $h(x), h(y)$ are independent for any two items $x \neq y$. 

Caveat 1: It is very expensive to represent and compute such a random function. We will see how a hash function computable in $O(1)$ time function can be used instead.

Caveat 2: In practice, often suffices to use hash functions like MD5, SHA-2, etc. that 'look random enough'.


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Let $C_{i,j} = 1$ if items $i$ and $j$ collide ($h(x_i) = h(x_j)$), and 0 otherwise. The number of pairwise duplicates is:

$$C = \sum_{i,j \in [m], i<j} C_{i,j}.$$ 

$x_i, x_j$: pair of stored items, $m$: total number of stored items, $n$: hash table size, $C$: total pairwise collisions in table, $h$: random hash function.
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$$E[C] = \sum_{i,j \in [m], i < j} \frac{1}{n} = \frac{(m)}{2} = \frac{m(m - 1)}{2n}.$$

$x_i, x_j$: pair of stored items, $m$: total number of stored items, $n$: hash table size, $C$: total pairwise collisions in table, $h$: random hash function.
Let $C_{i,j} = 1$ if items $i$ and $j$ collide ($h(x_i) = h(x_j)$), and 0 otherwise. The number of pairwise duplicates is:

$$
\mathbb{E}[C] = \sum_{i,j \in [m], i<j} \mathbb{E}[C_{i,j}].
$$

(linearity of expectation)

For any pair $i, j$, $i < j$:

$$
\mathbb{E}[C_{i,j}] = \Pr[C_{i,j} = 1] = \Pr[h(x_i) = h(x_j)] = \frac{1}{n}.
$$

$$
\mathbb{E}[C] = \sum_{i,j \in [m], i<j} \frac{1}{n} = \frac{\binom{m}{2}}{n} = \frac{m(m-1)}{2n}.
$$

Identical to the CAPTCHA analysis!

$x_i, x_j$: pair of stored items, $m$: total number of stored items, $n$: hash table size, $C$: total pairwise collisions in table, $h$: random hash function.
\[ E[C] = \frac{m(m-1)}{2n}. \]

\hspace{1cm}

\[ m: \text{ total number of stored items}, \ n: \text{ hash table size}, \ C: \text{ total pairwise collisions in table}. \]
\[ \mathbb{E}[C] = \frac{m(m-1)}{2n}. \]

- For \( n = 4m^2 \) we have: \( \mathbb{E}[C] = \frac{m(m-1)}{8m^2} \leq \frac{1}{8}. \)

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Collision Free Hashing

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Apply Markov’s Inequality:

\[ \Pr[C \geq 1] \leq \frac{\mathbb{E}[C]}{1} = \frac{1}{8} . \]

\[ \Pr[C = 0] = 1 - \Pr[C \geq 1] \geq 1 - \frac{1}{8} = \frac{7}{8} . \]

Pretty good but we are using \( O(m^2) \) space to store \( m \) items.

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\end{align*}
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Pretty good but we are using \( O(m^2) \) space to store \( m \) items.

\( m \): total number of stored items, \( n \): hash table size, \( C \): total pairwise collisions in table.
Want to preserve $O(1)$ query time while using $O(m)$ space.
Two-Level Hashing:

Want to preserve $O(1)$ query time while using $O(m)$ space.

For each bucket with $s_i$ values, pick a collision free hash function mapping $\{s_i\} \rightarrow \{4s_i^2\}$.

Just showed: A random function is collision free with probability $\geq \frac{7}{8}$, so only requires checking $O(1)$ random functions in expectation to find a collision free one.
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Query time for two level hashing is $O(1)$: requires evaluating two hash functions.

$x_j, x_k$: stored items, $n$: hash table size, $h$: random hash function, $S$: space usage of two level hashing, $s_i$: # items stored in hash table at position $i$. 
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Query time for two level hashing is $O(1)$: requires evaluating two hash functions. *What is the expected space usage?*

Up to constants, space used is: $S = n + 4 \sum_{i=1}^{n} s_i^2$

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\]

Collisions again!

\(x_j, x_k\): stored items, \(n\): hash table size, \(h\): random hash function, \(S\): space usage of two level hashing, \(s_i\): \# items stored in hash table at position \(i\).
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$$= E \left[ \sum_{j, k \in [m]} \mathbb{1}_{h(x_j) = i} \cdot \mathbb{1}_{h(x_k) = i} \right] = \sum_{j, k \in [m]} E \left[ \mathbb{1}_{h(x_j) = i} \cdot \mathbb{1}_{h(x_k) = i} \right].$$

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• For $j = k$,

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$$
\mathbb{E}[s_i^2] = \mathbb{E} \left[ \left( \sum_{j=1}^{m} I_{h(x_j)=i} \right)^2 \right] \\
= \mathbb{E} \left[ \sum_{j,k \in [m]} I_{h(x_j)=i} \cdot I_{h(x_k)=i} \right] \\
= \sum_{j,k \in [m]} \mathbb{E} \left[ I_{h(x_j)=i} \cdot I_{h(x_k)=i} \right] .
$$

- For $j = k$, $\mathbb{E} \left[ I_{h(x_j)=i} \cdot I_{h(x_k)=i} \right] = \mathbb{E} \left[ (I_{h(x_j)=i})^2 \right]$.

---

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$x_j, x_k$: stored items, $n$: hash table size, $h$: random hash function, $S$: space usage of two level hashing, $s_i$: # items stored in hash table at position $i$. 
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Space usage

\[ \mathbb{E}[s_i^2] = \sum_{j,k \in [m]} \mathbb{E} \left[ \mathbb{I}_{h(x_j) = i} \cdot \mathbb{I}_{h(x_k) = i} \right] = m \cdot \frac{1}{n} + 2 \cdot \left( \frac{m}{2} \right) \cdot \frac{1}{n^2} \]

- For \( j = k \), \( \mathbb{E} \left[ \mathbb{I}_{h(x_j) = i} \cdot \mathbb{I}_{h(x_k) = i} \right] = \frac{1}{n} \).
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\[
E[S_i^2] = \sum_{j,k \in [m]} E[I_h(x_j)=i \cdot I_h(x_k)=i]
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\]
\[
= m \cdot \frac{1}{n} + 2 \cdot \binom{m}{2} \cdot \frac{1}{n^2}
\]
\[
= \frac{m}{n} + \frac{m(m-1)}{n^2}
\]

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\[ \mathbb{E}[s_i^2] = \sum_{j,k \in [m]} \mathbb{E} \left[ \mathbb{I}_h(x_j) = i \cdot \mathbb{I}_h(x_k) = i \right] \]

\[ = m \cdot \frac{1}{n} + 2 \cdot \binom{m}{2} \cdot \frac{1}{n^2} \]

\[ = \frac{m}{n} + \frac{m(m-1)}{n^2} \leq 2 \text{ (If we set } n = m.\) }

- For \( j = k \), \( \mathbb{E} \left[ \mathbb{I}_h(x_j) = i \cdot \mathbb{I}_h(x_k) = i \right] = \frac{1}{n} \).
- For \( j \neq k \), \( \mathbb{E} \left[ \mathbb{I}_h(x_j) = i \cdot \mathbb{I}_h(x_k) = i \right] = \frac{1}{n^2} \).

\( x_j, x_k \): stored items, \( m \): \# stored items, \( n \): hash table size, \( h \): random hash function, \( S \): space usage of two level hashing, \( s_i \): \# items stored at pos \( i \).
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\mathbb{E}[s_i^2] = \sum_{j,k \in [m]} \mathbb{E}[\mathbb{1}_{h(x_j) = i} \cdot \mathbb{1}_{h(x_k) = i}]
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= m \cdot \frac{1}{n} + 2 \cdot \binom{m}{2} \cdot \frac{1}{n^2}
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**Total Expected Space Usage:** (if we set \( n = m \))

\[
\mathbb{E}[S] = n + 4 \sum_{i=1}^{n} \mathbb{E}[s_i^2]
\]

\( x_j, x_k \): stored items, \( m \): \# stored items, \( n \): hash table size, \( h \): random hash function, \( S \): space usage of two level hashing, \( s_i \): \# items stored at pos \( i \).
\[ \mathbb{E}[\mathbf{s}_i^2] = \sum_{j,k \in [m]} \mathbb{E} \left[ I_h(x_j) = i \cdot I_h(x_k) = i \right] \]
\[ = m \cdot \frac{1}{n} + 2 \cdot \binom{m}{2} \cdot \frac{1}{n^2} \]
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**Total Expected Space Usage:** (if we set \( n = m \))

\[ \mathbb{E}[S] = n + 4 \sum_{i=1}^{n} \mathbb{E}[s_i^2] \leq n + 4n \cdot 2 = 9n = 9m. \]

\( x_j, x_k \): stored items, \( m \): \# stored items, \( n \): hash table size, \( h \): random hash function, \( S \): space usage of two level hashing, \( s_i \): \# items stored at pos \( i \).
\[ E[S_i^2] = \sum_{j, k \in [m]} E[\mathbb{I}_{h(x_j) = i} \cdot \mathbb{I}_{h(x_k) = i}] \]
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**Total Expected Space Usage:** (if we set \( n = m \))
\[ E[S] = n + 4 \sum_{i=1}^{n} E[S_i^2] \leq n + 4n \cdot 2 = 9n = 9m. \]

Near optimal space with \( O(1) \) query time!

\( x_j, x_k \): stored items, \( m \): \# stored items, \( n \): hash table size, \( h \): random hash function, \( S \): space usage of two level hashing, \( s_i \): \# items stored at pos \( i \).