Today:

• Investigate linearity of expectation and variance.
• Algorithmic application of linearity of expectation and variance.
• Introduce Markov’s inequality, a fundamental concentration bound, that let us prove that a random variable lies close to its expectation with good probability.
• Learn about random hash functions, which are a key tool in randomized methods for data processing. Probabilistic analysis via linearity of expectation.
• **Expectation:**  
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• **Variance:** \[ \text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]. \]
• **Expectation:** \( \mathbb{E}[X] = \sum_{s \in S} \Pr(X = s) \cdot s \).

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• Two random variables \( X, Y \) are **independent** if for all \( s, t \), \( \{X = s\} \) and \( \{Y = t\} \) are independent events. In other words:

\[
\Pr(\{X = s\} \cap \{Y = t\}) = \Pr(X = s) \cdot \Pr(Y = t).
\]
When are the expectation and variance linear?

I.e., under what conditions on $X$ and $Y$ do we have:

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

and

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y].$$

$X, Y$: any two random variables.
LINEARITY OF EXPECTATION

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Proof:
**LINEARITY OF EXPECTATION**

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**Proof:**

\[ \mathbb{E}[X + Y] = \sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot (s + t) \]
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(law of total probability)
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Linearity of Variance

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- They claim that they have a database of 1,000,000 unique CAPTCHAS. A random one is chosen for each security check.
- You want to independently verify this claimed database size.
- You could make test checks until you see 1,000,000 unique CAPTCHAS: would take \( \geq 1,000,000 \) checks!
An Idea: You run some test security checks and see if any duplicate CAPTCHAS show up. If you’re seeing duplicates after not too many checks, the database size is probably not too big.
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An Idea: You run some test security checks and see if any **duplicate CAPTCHAS** show up. If you’re seeing duplicates after not too many checks, the database size is probably not too big.

‘Mark and recapture’ method in ecology.

Note that if the same CAPTCHA shows up four times this counts as \( \binom{4}{2} \) duplicates.
Let $D_{i,j} = 1$ if tests $i$ and $j$ give the same CAPTCHA, and 0 otherwise. An indicator random variable.

\[ n: \text{number of CAPTCHAs in database}, \quad m: \text{number of random CAPTCHAs drawn to check database size}, \quad D: \text{number of pairwise duplicates in } m \text{ random CAPTCHAS} \]
Let $D_{i,j} = 1$ if tests $i$ and $j$ give the same CAPTCHA, and 0 otherwise. An indicator random variable.

\begin{align*}
D_{1,2} &= 0 \\
D_{2,4} &= 1
\end{align*}

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Let $D_{i,j} = 1$ if tests $i$ and $j$ give the same CAPTCHA, and 0 otherwise. An indicator random variable. The number of pairwise duplicates (a random variable) is:

$$D = \sum_{i,j \in [m], i \neq j} D_{i,j}.$$ 

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For any pair $i, j \in [m], i \neq j$: $\mathbb{E}[D_{i,j}] = \Pr[D_{i,j} = 1] = \frac{1}{n}.$
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You take $m = 1000$ samples. If the database size is as claimed ($n = 1,000,000$) then expected number of duplicates is:

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**Concentration Inequalities:** Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.
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For any **non-negative** random variable $X$ and any $t > 0$:

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The larger the deviation $t$, the smaller the probability.
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Applying Markov’s inequality, if the real database size is \( n = 1,000,000 \) the probability of this happening is:

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\Pr[D \geq 10] \leq \frac{\mathbb{E}[D]}{10} = \frac{0.4995}{10} \approx 0.05
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This is pretty small – you feel pretty sure the number of unique CAPTCHAS is much less than 1,000,000. But how can you boost your confidence?

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**Classic Solution:** Hash tables

- *Static hashing* since we won’t worry about insertion and deletion today.
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• **Collisions:** when we insert $m$ items into the hash table we may have to store multiple items in the same location (typically as a linked list).
Query runtime: \( O(c) \) when the maximum number of collisions in a table entry is \( c \) (i.e., must traverse a linked list of size \( c \)).
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How Can We Bound $c$?

- In the worst case, could have $c = m$ (all items hash to the same location). In the best case, $c \approx m/n$. 
Let $h : U \to [n]$ be a random hash function.

- i.e., for $x \in U$, $\Pr(h(x) = i) = \frac{1}{n}$ for all $i = 1, \ldots, n$ and $h(x), h(y)$ are independent for any two items $x \neq y$. 

Caveat 1: It is very expensive to represent and compute such a random function. We will see how a hash function computable in $O(1)$ time function can be used instead.

Caveat 2: In practice, often suffices to use hash functions like MD5, SHA-2, etc. that 'look random enough'. 
Let $h : U \rightarrow [n]$ be a random hash function.

- I.e., for $x \in U$, $\Pr(h(x) = i) = \frac{1}{n}$ for all $i = 1, \ldots, n$ and $h(x), h(y)$ are independent for any two items $x \neq y$.
- **Caveat 1:** It is very expensive to represent and compute such a random function. We will see how a hash function computable in $O(1)$ time function can be used instead.
- **Caveat 2:** In practice, often suffices to use hash functions like MD5, SHA-2, etc. that ‘look random enough’.
Let $C_{i,j} = 1$ if items $i$ and $j$ collide ($h(x_i) = h(x_j)$), and 0 otherwise. The number of pairwise duplicates is:

$$C = \sum_{i,j \in [m], i \neq j} C_{i,j}.$$ 

$x_i, x_j$: pair of stored items, $m$: total number of stored items, $n$: hash table size, $C$: total pairwise collisions in table, $h$: random hash function.
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Identical to the CAPTCHA analysis!

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Pretty good...but we are using \( O(m^2) \) space to store \( m \) items...

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- For each bucket with $s_i$ values, pick a collision free hash function mapping $[s_i] \rightarrow [s_i^2]$.

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![Diagram showing two-level hashing]

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SPACE USAGE

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Collisions again!

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\[ = m \cdot \frac{1}{n} + 2 \cdot \left( \begin{array}{c} m \\ 2 \end{array} \right) \cdot \frac{1}{n^2} \]

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Near optimal space with \( O(1) \) query time!

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**Can we use even smaller space?**

**Many Applications:**

- Filter spam email addresses, phone numbers, suspect IPs, duplicate Tweets.
- Quickly check if an item has been stored in a cache or is new.
- Counting distinct elements (e.g., unique search queries.)
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**Efficient Alternative:** Let $p$ be a prime with $p \geq |U|$. Choose random $a, b \in [p]$ with $a \neq 0$. Let:

$$h(x) = (ax + b \mod p) \mod n.$$
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Questions on linearity of expectation, Markov’s, hashing?
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2. Then we’ll show how a simple twist on Markov’s can give a much stronger result.
Randomized Load Balancing:

Another application

Simple Model: \( n \) requests randomly assigned to \( k \) servers. How many requests must each server handle?
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Expected Number of requests assigned to server $i$:

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WEAKNESS OF MARKOV’S

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Not great...half the servers may be overloaded.

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$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}[X]}{t^2}.$$  

(by plugging in the random variable $X - \mathbb{E}[X]$)
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**X:** any random variable, **t, s:** any fixed numbers.
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Why is this so powerful?

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$$\text{Var}[S] = \frac{1}{n^2} \text{Var} \left[ \sum_{i=1}^{n} X_i \right]$$

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With enough samples $n$, the sample average will always concentrate to the mean $\mu$.

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$$\Pr(\left| S - \mathbb{E}[S] \right| \geq \epsilon) \leq \frac{\text{Var}[S]}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}.$$
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**Chernoff Type Bounds:** A quantitative version of the central limit theorem. The average of many independent random variables is distributed like a Gaussian.