Spectral Graph Partitioning

• Focus on separating graphs with small but relatively balanced cuts.
• Connection to second smallest eigenvector of graph Laplacian.
• Today: Provable guarantees for stochastic block model.
To partition a graph, find the eigenvector of the Laplacian with the second smallest eigenvalue. Partition nodes based on whether corresponding value in eigenvector is positive/negative.

\[
\mathbf{v}_{n-1} = \arg \min_{\mathbf{v} \in \mathbb{R}^n, \|\mathbf{v}\| = 1, \mathbf{v}^T \mathbf{I} = 0} \mathbf{v}^T L \mathbf{v}
\]

We argued this "should" partition graph along a small cut that separates the graph into large pieces.

Haven’t given formal guarantees; it’s difficult for general input graphs. . .

Common Approach:
Give a natural generative model for random inputs and analyze how the algorithm performs on inputs drawn from this model. Can be used to justify $\ell_2$ linear regression, $k$-means clustering, etc.
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\[ \vec{v}_{n-1} = \underset{\vec{v} \in \mathbb{R}^n, \|\vec{v}\|=1, \vec{v}^T \vec{1}=0}{\arg \min} \vec{v}^T \mathcal{L} \vec{v} \]

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Common Approach: Give a natural generative model for random inputs and analyze how the algorithm performs on inputs drawn from this model. Can be used to justify \( \ell_2 \) linear regression, \( k \)-means clustering, etc.
Stochastic Block Model (Planted Partition Model): Let $G_n(p, q)$ be a distribution over graphs on $n$ nodes, split randomly into two groups $B$ and $C$, each with $n/2$ nodes.

- Any two nodes in the **same group** are connected with probability $p$ (including self-loops).
- Any two nodes in **different groups** are connected with prob. $q < p$.
- Connections are independent.
Let $G$ be a stochastic block model graph drawn from $G_n(p, q)$.

- Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix of $G$, ordered in terms of group ID.

$G_n(p, q)$: stochastic block model distribution. $B, C$: groups with $n/2$ nodes each. Connections are independent with probability $p$ between nodes in the same group, and probability $q$ between nodes not in the same group.
Letting $G$ be a stochastic block model graph drawn from $G_n(p, q)$ and $A \in \mathbb{R}^{n \times n}$ be its adjacency matrix. $(\mathbb{E}[A])_{i,j} = p$ for $i,j$ in same group, $(\mathbb{E}[A])_{i,j} = q$ otherwise.

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What is $\text{rank}(\mathbb{E}[A])$? What are the eigenvectors and eigenvalues of $\mathbb{E}[A]$?

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- Can show that for $G \sim G_n(p, q)$, $A$ is “close” to $E[A]$ in some appropriate sense (matrix concentration inequality).
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- Can show that for $G \sim G_n(p, q)$, $A$ is “close” to $E[A]$ in some appropriate sense (matrix concentration inequality).
- Second eigenvector of $A$ is close to $[1, 1, 1, \ldots, -1, -1, -1]$ and gives a good estimate of the communities.
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- Second eigenvector of $A$ is close to $[1, 1, 1, \ldots, -1, -1, -1]$ and gives a good estimate of the communities.

When rows/columns aren’t sorted by ID, second eigenvector is e.g., $[1, -1, 1, -1, \ldots, 1, 1, -1]$ and entries give community ids.
Letting $G$ be a stochastic block model graph drawn from $G_n(p, q)$, $A \in \mathbb{R}^{n \times n}$ be its adjacency matrix and $L$ be its Laplacian, what are the eigenvectors and eigenvalues of $\mathbb{E}[L]$?
Letting $G$ be a stochastic block model graph drawn from $G_n(p, q)$, $A \in \mathbb{R}^{n \times n}$ be its adjacency matrix and $L$ be its Laplacian, what are the eigenvectors and eigenvalues of $\mathbb{E}[L]$?

$$\mathbb{E}[L] = \mathbb{E}[D] - \mathbb{E}[A] = \left(\frac{n(p + q)}{2}\right) I - \mathbb{E}[A]$$

and so if $\mathbb{E}[A] \vec{x} = \lambda \vec{x}$ then

$$\mathbb{E}[L] \vec{x} = (n(p + q)/2 - \lambda) \vec{x}$$
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Therefore the first and second eigenvalues of $\mathbb{E}[A]$ are the second and first eigenvectors of $\mathbb{E}[L]$. 

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- If the matrices $\mathbf{A}$ and $\mathbf{L}$ were exactly equal to their expectation, partitioning using this eigenvector (i.e., spectral clustering) would exactly recover the two communities $B$ and $C$.

**EXPECTED LAPLACIAN SPECTRUM**
Upshot: The second smallest eigenvector of $E[L]$ is $\chi_{B,C}$ – the indicator vector for the cut between the communities.

- If the matrices $A$ and $L$ were exactly equal to their expectation, partitioning using this eigenvector (i.e., spectral clustering) would exactly recover the two communities $B$ and $C$.

How do we show that a matrix is close to its expectation? Matrix concentration inequalities.

- Analogous to scalar concentration inequalities like Markovs, Chebyshevs, Bernsteins.
- Random matrix theory is a very recent and cutting edge subfield of mathematics that is being actively applied in computer science, statistics, and ML.
Matrix Concentration Inequality: If $p \geq O\left(\frac{\log^4 n}{n}\right)$, then with high probability

$$\|A - \mathbb{E}[A]\|_2 \leq O(\sqrt{pn}).$$

where $\|\cdot\|_2$ is the matrix spectral norm (operator norm).

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For the stochastic block model application, we want to show that the second eigenvectors of \( A \) and \( \mathbb{E}[A] \) are close. How does this relate to their difference in spectral norm?
Davis-Kahan Eigenvector Perturbation Theorem: Suppose $A, \overline{A} \in \mathbb{R}^{d \times d}$ are symmetric with $\|A - \overline{A}\|_2 \leq \epsilon$ and eigenvectors $v_1, v_2, \ldots, v_d$ and $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_d$. Letting $\theta(v_i, \overline{v}_i)$ denote the angle between $v_i$ and $\overline{v}_i$, for all $i$:

$$\sin[\theta(v_i, \overline{v}_i)] \leq \frac{\epsilon}{\min_{j \neq i} |\lambda_i - \lambda_j|}$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of $\overline{A}$.

The errors get large if there’s eigenvalues with similar magnitudes.
Claim 1 (Matrix Concentration): For $p \geq O\left(\frac{\log^4 n}{n}\right)$,

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Claim 2 (Davis-Kahan): For $p \geq O\left(\frac{\log^4 n}{n}\right)$,

$$\sin \theta(v_2, \bar{v}_2) \leq \frac{O(\sqrt{pn})}{\min_{j \neq 2} |\lambda_2 - \lambda_j|}.$$
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Recall: $\mathbb{E}[A]$ has eigenvalues $\lambda_1 = \frac{(p+q)n}{2}, \lambda_2 = \frac{(p-q)n}{2}, \lambda_i = 0$ for $i \geq 3$. 

A adjacency matrix of random stochastic block model graph. $p$: connection probability within clusters. $q < p$: connection probability between clusters. $n$: number of nodes. $v_2, \bar{v}_2$: second eigenvectors of $A$ and $\mathbb{E}[A]$ respectively.
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$$\min_{j \neq 2} |\lambda_2 - \lambda_j| = \min \left( qn, \frac{(p-q)n}{2} \right).$$

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Typically, $\frac{(p-q)n}{2}$ will be the minimum of these two gaps.

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$$\sin \theta(v_2, \bar{v}_2) \leq \frac{O(\sqrt{pn})}{\min_{j \neq 2} |\lambda_2 - \lambda_j|} \leq \frac{O(\sqrt{pn})}{(p - q)n/2} = O\left(\frac{\sqrt{p}}{(p - q)\sqrt{n}}\right)$$

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- Can show that this implies \( \| \mathbf{v}_2 - \bar{\mathbf{v}}_2 \|_2^2 \leq O \left( \frac{p}{(p-q)^2 n} \right) \) (exercise).

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- \( \bar{v}_2 \) is \( \frac{1}{\sqrt{n}} \chi_{B,C} \): the community indicator vector.

\[ \begin{array}{cccccccc}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n}} \\
\end{array} \]

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APPLICATION TO STOCHASTIC BLOCK MODEL

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\begin{align*}
B \quad & \quad \quad C \\
\frac{1}{\sqrt{n}} & \quad \frac{1}{\sqrt{n}} & \quad \frac{1}{\sqrt{n}} & \quad \frac{1}{\sqrt{n}} & \quad \frac{1}{\sqrt{n}} & \quad \frac{1}{\sqrt{n}} & \quad \frac{1}{\sqrt{n}} & \quad \frac{1}{\sqrt{n}} \\
\bar{v}_2
\end{align*}

- Every \( i \) where \( v_2(i), \bar{v}_2(i) \) differ in sign contributes \( \geq \frac{1}{n} \) to \( \|v_2 - \bar{v}_2\|_2^2 \).

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**APPLICATION TO STOCHASTIC BLOCK MODEL**

**So Far:** \( \sin \theta(v_2, \bar{v}_2) \leq O\left(\frac{\sqrt{p}}{(p-q)^2} \frac{1}{\sqrt{n}} \right) \). What does this give us?

- Can show that this implies \( \|v_2 - \bar{v}_2\|^2 \leq O\left(\frac{p}{(p-q)^2 n} \right) \) (exercise).
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- Every \( i \) where \( v_2(i), \bar{v}_2(i) \) differ in sign contributes \( \geq \frac{1}{n} \) to \( \|v_2 - \bar{v}_2\|^2 \).
- So they differ in sign in at most \( O\left(\frac{p}{(p-q)^2} \right) \) positions.

\[ \begin{array}{cccccccc}
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \end{array} \]

\( \bar{v}_2 \)

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\( v_2, \bar{v}_2 \): second eigenvectors of **A** and \( \mathbb{E}[\mathbf{A}] \) respectively.
**Upshot:** If $G$ is a stochastic block model graph with adjacency matrix $A$, if we compute its second large eigenvector $v_2$ and assign nodes to communities according to the sign pattern of this vector, we will correctly assign all but $O\left(\frac{p}{(p-q)^2}\right)$ nodes.